

MONTHLY NOTICES
OF THE
ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1937 MAY 14

Dr. H. Spencer Jones, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Jiri Alter, University Observatory, 14 Čechova, Praha VII, Czechoslovakia ;

Donald Robert Barber, B.Sc., A.Inst.P., Lyndhurst, Salcombe Road, Sidmouth, Devon ;

David Stanley Evans, King's College, Cambridge ; and

Theodore E. Sterne, Ph.D., Harvard College Observatory, Cambridge, Mass.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

William Bradshaw Housman, The Observatory, Seaton, Workington, Cumberland (proposed by T. E. R. Phillips) ;

Frederick George Jones, 8 Clifton Vale, Clifton, Bristol (proposed by J. H. H. Robinson) ; and

James Barrett Livingston, Warwick Close, Weansway, Bishop's Avenue, London, N. 2 (proposed by E. W. Barlow).

One hundred and eleven presents were announced as having been received since the last ordinary meeting.

The President announced that the late Miss Julia Lindley had bequeathed to the Society the sum of £1000.



Professor N. E. Nörlund delivered the George Darwin Lecture, taking as his subject "Astronomical Longitude and Azimuth Determinations" (see p. 489).

Professor Count F. M. da Costa Lobo spoke on some researches carried out by him at the Cöimbra Observatory.

The following Geophysical Meetings have been held during the present session :—

- 1936 Oct. 20. The Gaseous Content of the Atmosphere at various Altitudes. Opener, Prof. Paneth.
 Nov. 27. Reading and Discussion of Papers presented to the Society.
 1937 Jan. 29. Terrestrial Heights. Opener, Dr. J. de Graaff Hunter.
 Feb. 26. Radio Observations in High Latitudes. Opener, Prof. E. V. Appleton.
 Mar. 19. Instrumental Seismology. Opener, Mr. J. J. Shaw.

CELESTIAL PHOTOGRAPHS.

The following have been added to the list of Celestial Photographs reproduced by the Society for sale to Fellows:—

R.A.S.

Serial No.	Ref. No.	
494	A 40	Solar Corona, 1936 June 19. R. L. Waterfield, Chios. 6-inch O.G., <i>f.l.</i> 12 feet 6 inches. Exp. 15 sec.
495	A 41	Solar Corona, 1936 June 19. R. L. Waterfield, Chios. 6-inch O.G., <i>f.l.</i> 12 feet 6 inches. Exp. 1 sec.
496	L 37	Milky Way, Cassiopeia, Andromeda α , $0^h 36^m$, δ , $+49^\circ$. F. E. Ross, Yerkes. 5-inch lens, <i>f.l.</i> 35 inches. 1930 October 16. Exp. 3 h.
497	L 38	Milky Way, Cassiopeia, Perseus α , $1^h 37^m$, δ , $+63^\circ$. F. E. Ross, Yerkes. 5-inch lens, <i>f.l.</i> 35 inches. 1934 September 6-7. Exp. 3 h.
498	L 39	Milky Way, Perseus, Auriga α , $4^h 42^m$, δ , $+32^\circ$. F. E. Ross, Yerkes. 5-inch lens, <i>f.l.</i> 35 inches. 1935 September 21-22. Exp. 4 h.
499	L 40	Milky Way, Orion α , $5^h 26^m$, δ , -1° . F. E. Ross, Yerkes. 5-inch lens, <i>f.l.</i> 35 inches. 1935 October 5-6. Exp. 4 h.
500	J 96	Nebula I.C. 1613, Cetus. Baade, Mt. Wilson. 100-inch refl., <i>f.l.</i> 504 inches. 1933 November 14.
501	J 97	N.G.C. 3185, '87, '90, '93, Leo. Mt. Wilson. 100-inch refl., <i>f.l.</i> 504 inches. 1935 December 24.

ASTRONOMICAL LONGITUDE AND AZIMUTH DETERMINATIONS.

(*George Darwin Lecture, delivered by Dr. N. E. Nörlund, Assoc.R.A.S.,
on 1937 May 14.*)

I am afraid that the subject I have chosen for this lecture will not be considered very attractive. In the later decennia the attention of the astronomers has to a large extent been directed towards astrophysical problems which have led to many new and interesting results. Under these circumstances an old classic problem like the determination of longitude has naturally fallen a little into the background.

The determination of longitudes is one of the most difficult astronomical operations. It demands great experience and care. The various sources of error have been examined so carefully that it will hardly be possible to add any essential aspects. As, however, the problem is of fundamental importance, especially because geodesy needs an increasing number of exact longitudes, even a small contribution to the elucidation of this matter may be considered of interest.

After the introduction of the electric telegraph a considerable number of longitude determinations were carried out in the course of the nineteenth century. But as the stations had to be connected through wires or cables, long diplomatic negotiations were often a necessary preliminary to the observations. The most important determinations had therefore to suffice, and it was impossible to measure as many stations as was desirable for a connection of the initial points of all systems of triangulation and for a determination of the deviations of the vertical. Also, the precision obtained was not always as great as might have been expected from the agreement between the individual observations. The history of the determination of the longitude Paris-Greenwich, with the great discordance between the independent determinations of the English and French observers, is well known. Since the carrying out of these measurements the determination of longitudes has been facilitated, thanks to the wireless time-signals which have completely altered the problems of longitude determination. We are no longer concerned with the routes followed by land lines and cables, and a more effective control of the various sources of error has been made possible. In several papers published in the *Monthly Notices of the Royal Astronomical Society* about fifteen years ago, Professor Sampson and Sir Frank Dyson called attention to the discrepancies in the time determinations of some of the principal observatories. The times of reception of the time-signals generally agree closely for successive days, but are systematically different over extended periods. For the most part the fluctuations appear to depend on the time determinations at the observatories. These very peculiar discordances were further illustrated by the work of the Bureau de l'Heure and by the

international longitude determinations in 1926 and 1933. But longitude determinations are still encumbered with very considerable systematic errors of unknown origin, which require a further careful examination of all possible sources of error.

A question of a certain interest is which stars should be preferred for a longitude or an azimuth determination. The instructions given by the various observatories and geodetic institutes for the selection of the stars are rather different. Some recommend the balance method, others prefer the method of observing equatorial stars combined with close circumpolar stars. But the selection of stars is not without influence on the result. If, for instance, at 60° latitude we make a time determination by observing 1 close circumpolar star and 11 equatorial stars, the weight will be 8 times smaller than for the best selection of 12 stars, and even 3 stars with a small declination give nearly the same weight as 11, so that 8 observations have been almost wasted. In the International Geodetic Association the question of the best selection of stars has recently been raised by K. Wold,* and F. Björnseth and J. Schive† have published several papers on this subject especially considering the high latitudes. The problem has further been treated by B. Aurell‡ and Th. Niethammer.§ The following statement is related to that of Niethammer, who treats the case where the parameter ρ referred to below is zero.

Longitude determinations are best carried out by means of a portable transit which is reversed at the middle of each observation, and I will confine myself to considering this case. Let t and t' denote the observed times of transit of two stars with the right ascensions α and α' . If we put

$$l = \alpha - t, \quad l' = \alpha' - t',$$

and call the inclination i , the clock correction u and the azimuth k will be determined by the well-known equations

$$\begin{aligned} u + kK + i\mathfrak{J} &= l, \\ u + kK' + i\mathfrak{J}' &= l', \end{aligned} \tag{1}$$

where

$$K = \frac{\sin z}{\sin p}, \quad \mathfrak{J} = \frac{\cos z}{\sin p},$$

and z is the zenith distance reckoned positive towards the south, while p is the polar distance reckoned positive in the upper and negative in the lower culmination. From equation (1) we derive

$$u = \frac{Kl' - K'l}{K - K'} - i \sec \phi, \tag{2}$$

$$k = \frac{l - l'}{K - K'} + i \tan \phi, \tag{3}$$

* *B. Geod.*, 14, 92, 1927.

† *B. Geod.*, 18, 370, 1928; 25, 1, 1930; 42, 80, 1934.

‡ *Bestimmung der Längendifferenz zwischen den astronomisch-geodätischen Zentralen Schwedens und Dänemarks*, Stok., 1933.

§ *A.N.*, 252, 205, 1934.

where ϕ is the latitude. The mean error of the observed clock-time of the meridian passage is, according to Albrecht, of the form

$$m_t^2 = a^2 + b^2 \sec^2 \delta, \quad (4)$$

where δ is the declination of the star and a and b are constants. Let us consider the ratio between the constants a and b ,

$$\rho = \frac{a}{b}.$$

From the determinations of longitudes made by the Danish Geodetic Institute in the years 1925-1933 I have derived the following averages for 7 observers :—

Moving wire micrometer worked by motor,	1.0.
Moving wire micrometer worked by hand,	1.3.

From the measurements of the Prussian Geodetic Institute Albrecht has derived the following values, corresponding to an enlargement of one hundred times :—

Eye and ear method,	2.1.
Key and chronograph method,	1.5.
Moving wire micrometer,	1.9.

Previous measurements published by the U.S. Coast and Geodetic Survey found $\rho = 1.4$. Accordingly, it seems likely that ρ lies between 1 and 2. The value of ρ is greatest in the case of the eye and ear method. In permanent observatories the moving wire micrometer will of course always be preferred, but at field stations in places difficult of access it may be necessary to use the eye and ear method. To give an example, the Finnish Geodetic Institute has in recent years carried out a great number of longitude determinations by means of the eye and ear method.

If provisionally we neglect the influence of the right ascension errors and the level error, equation (2) leads to the following mean error μ_u in the clock correction :—

$$\mu_u^2 = \frac{K^2 m_t'^2 + K'^2 m_t^2}{(K - K')^2}. \quad (5)$$

Suppose that the observed stars form a group of n stars with the average zenith distance z , and another group of n' stars with the average zenith distance z' . If we take the arithmetical mean of each of the two groups, letting

$$N = n + n', \quad \nu = \frac{n}{n'},$$

(5) gives us

$$\mu_u^2 = \frac{b^2 \sec^2 \phi}{N} f(z, z', \nu, \phi), \quad (6)$$

where

$$f(z, z', \nu, \phi) = f_1(z, z', \nu) + \rho^2 f_2(z, z', \nu, \phi) \quad (7)$$

and

$$f_1(z, z', \nu) = \frac{(1 + \nu) \sin^2 z + \left(1 + \frac{1}{\nu}\right) \sin^2 z'}{\sin^2 (z - z')},$$

$$f_2(z, z', \nu, \phi) = \frac{(1 + \nu) \sin^2 z \sin^2 \phi + \left(1 + \frac{1}{\nu}\right) \sin^2 z' \sin^2 \phi}{\sin^2 (z - z')}.$$

The first problem I am going to treat is the distribution of values of the function f . From this can easily be derived the selection of stars for which the mean errors of the clock correction deviate only very little from the most favourable value.

The function f depends on the two variables z and z' as well as on the parameters ν , ϕ and ρ . It varies to a considerable extent with ν , but only slowly with ϕ and ρ . If we know the shape of the function for a small number of values of ϕ and ρ we can easily interpolate its value for all ϕ and ρ . By means of a small number of figures we may therefore see at a glance the values of the function for any given latitude ϕ .

The function f remains unchanged when z is replaced by z' and ν by $\frac{1}{\nu}$. Thus it is sufficient to consider values of $\nu \geq 1$. The function f_2 satisfies the functional equation

$$f_2(z, z', \nu, \phi) = f_2\left(\phi - \frac{\pi}{2} - z', \phi - \frac{\pi}{2} - z, \nu, \phi\right).$$

If we put

$$\cot z = x, \quad \cot z' = y,$$

we readily realise that in the x, y plane the curve

$$f_2 = c$$

is a system of two straight lines, and the curve

$$f_1 = c$$

is a conic section which is easily constructed. We put

$$f(z, z', \nu, \phi) = (1 + \rho^2 \cos^2 \phi) g(z, z', \nu). \quad (8)$$

It is easy in the z, z' plane to construct the curve

$$g(z, z', \nu) = c, \quad (9)$$

where c is a constant. It is imaginary for $c < 1$ and real for $c > 1$. The smallest value which g can have is unity, and this value is reached only in the origin when a suitable approximation to this point is chosen. The origin is a singular point for g , in the immediate neighbourhood of which g can assume arbitrarily great values. This leads to the obvious result that the smallest value of the error is obtained by choosing all the stars exactly in zenith; this,

however, is a solution which can be neither realised nor recommended, it being unstable because the point is singular. Small deviations from zenith may cause arbitrarily great values of the error.

If we denote by W_N the weight of the clock correction in the case where a total number of N stars are observed, and put

$$W_N = \frac{a^2 + b^2 \sec^2 \phi}{\mu_u^2},$$

then we find

$$\frac{W_N}{N} = \frac{1}{g(z, z', \nu)}.$$

The function $g(z, z', \nu)$ is therefore the reciprocal weight divided by the number of stars. By constructing the curve (9) for various values of c we get a clear picture of the weight which may be obtained by any selection of stars. In the x, y plane the curve (9) is a hyperbola which can easily be represented in the z, z' plane. Each curve (for which $c > 1$) has a branch passing through the origin. Especially we have

$$g(z, 0, \nu) = 1 + \nu,$$

$$g(0, z', \nu) = 1 + \frac{1}{\nu}.$$

In that part of the z, z' plane lying between the ordinate axis and the curve corresponding to $c = 1 + \frac{1}{\nu}$ we have the inequality

$$g(z, z', \nu) < 1 + \frac{1}{\nu}.$$

Likewise the domain where

$$g(z, z', \nu) < 1 + \nu$$

is bounded by the abscissa axis and two branches of the curve corresponding to $c = 1 + \nu$. If $\nu = 1$ the two curves coincide, and the inequality

$$g(z, z', 1) < 2$$

is valid in two triangles which are bounded by the co-ordinate axes and two branches of the curve $g(z, z', 1) = 2$.

More generally we have

$$c_1 < g(z, z', \nu) < c_2$$

in the part of the plane which lies between the curves

$$g = c_1 \quad \text{and} \quad g = c_2.$$

The figures show for $\rho = 1$ a number of curves each corresponding to a constant value of the reciprocal weight indicated by the number written on the curve. Any point represents two groups of stars which are determined by means of their average zenith distances z and z' , and the curves indicate

the combinations of stars for which a given weight is obtained. In fig. 1 the latitude is 30° , and the ratio ν between the numbers of stars in the two groups

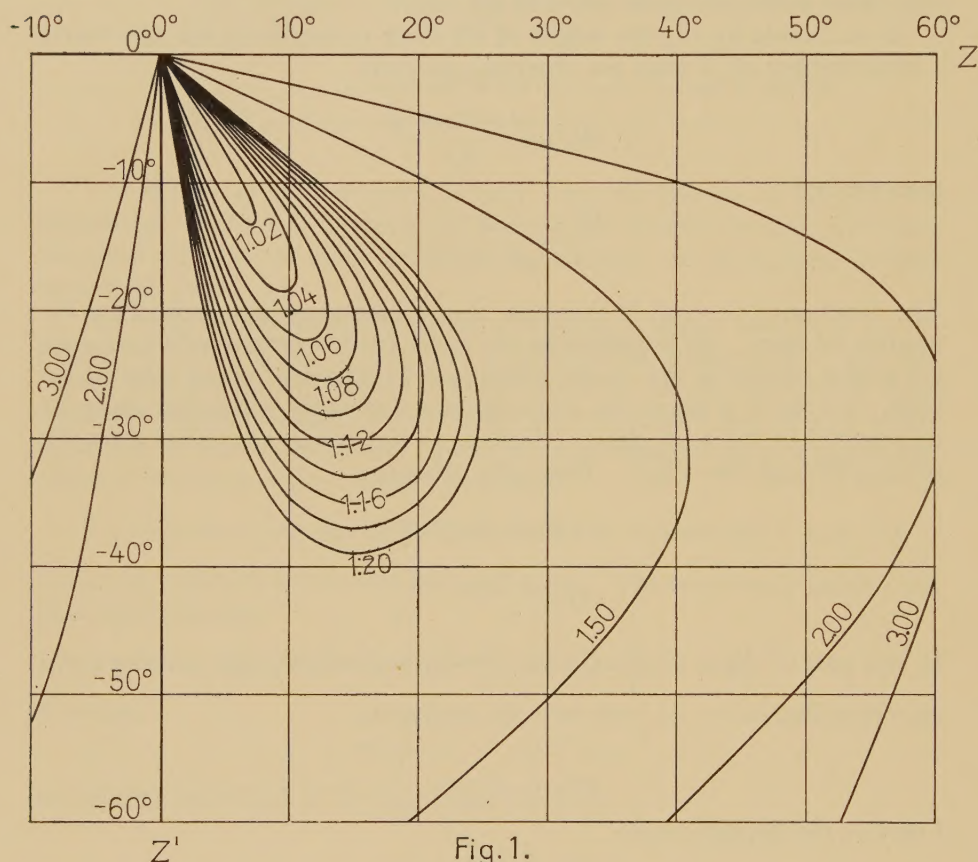


Fig. 1.

is equal to 2. The figure shows that a good weight is obtained by choosing, for instance, $z = +11^\circ$ and $z' = -20^\circ$. In fig. 2 the latitude is 45° and $\nu = 5$, and in fig. 3 the latitude is 75° and $\nu = 1$.

The curves show that it is advantageous to choose the clock stars at a rather small zenith distance.

If we vary z and keep z' constant, we find that $g(z, z', \nu)$ reaches a minimum for a value z_0 which is determined by the equation $\frac{\partial g}{\partial z} = 0$, or

$$\cot(z_0 - z') = -\frac{\nu}{\nu + 1} \cot z' + \frac{\rho^2}{\nu + 1} \frac{\tan(z' - \phi)}{\rho^2 + \sec^2(z' - \phi)}.$$

From this equation it is easy to tabulate z_0 as a function of z' and thus determine the most favourable zenith distance for the clock stars corresponding to any given azimuth star.

However, as there is only a small number of sufficiently bright stars at our disposal, the ideal conditions can never be exactly reached. For this reason

it is not sufficient to choose z in the neighbourhood of z_0 . It is also important that z' and ν be chosen so that a small change of z causes only a small

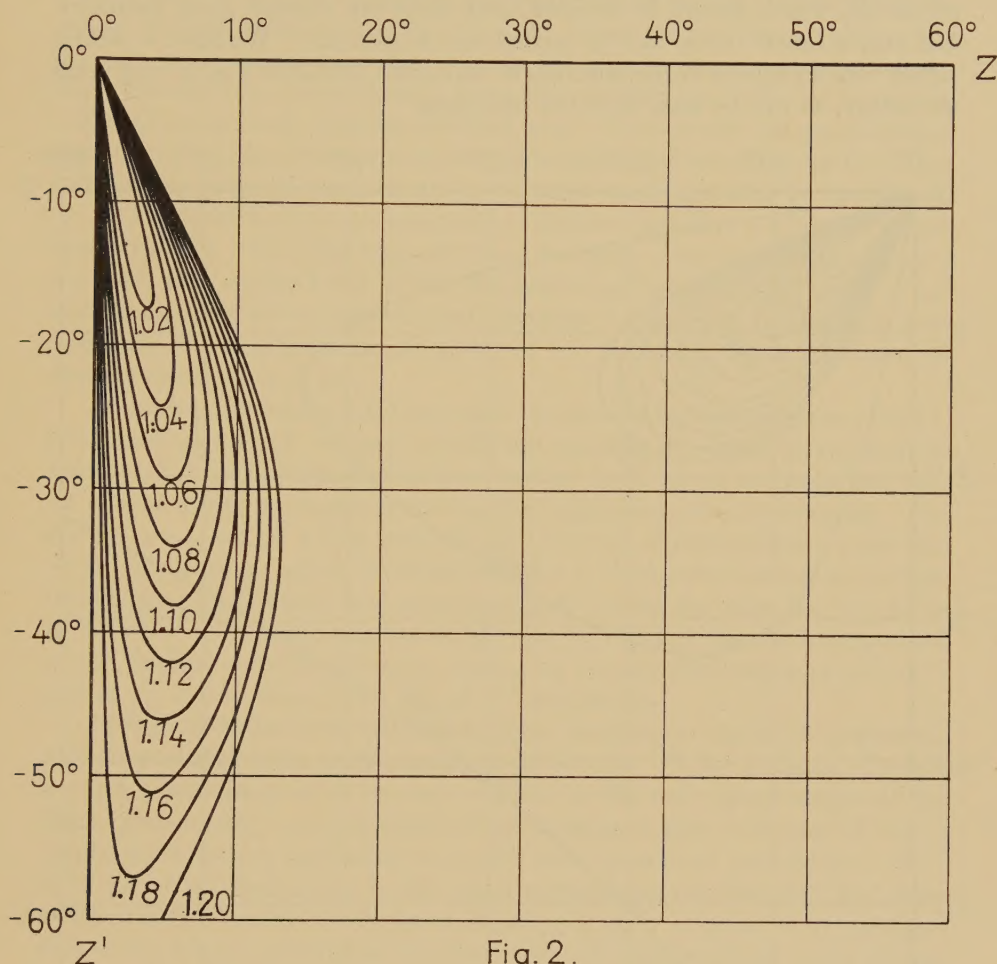


Fig. 2.

variation of $g(z, z', \nu)$. To secure this we consider the quantity $\frac{\partial^2 g}{\partial z'^2}$, which gives us the following approximate equation:—

$$g(z_0 + \epsilon, z', \nu) - g(z_0, z', \nu) = \epsilon^2 \frac{(1 + \nu) \left(1 + \frac{1}{\nu}\right)}{\sin^4(z_0 - z')} \sin^2 z' \frac{1 + \rho^2 \cos^2(z' - \phi)}{1 + \rho^2 \cos^2 \phi}.$$

This equation allows us to judge the stability of a chosen solution. We must avoid the domains where the weight curves lie close together, and give preference to domains where the greatest distance between the weight curves can be obtained. By this means we obtain a solution which ensures that a small deviation from the ideal value causes only a small loss of weight.

It appears from what has been said that the problem of the most favourable

selection of stars for the determination of the clock correction is one to which no very definite answer can be given. The figures show clearly the regions of the sky which should be avoided, and there are various good solutions, but only a few of them can be carried out in practice. We have a certain rather limited liberty in the selection of stars, and this liberty is of very great advantage, as will be seen from the following.

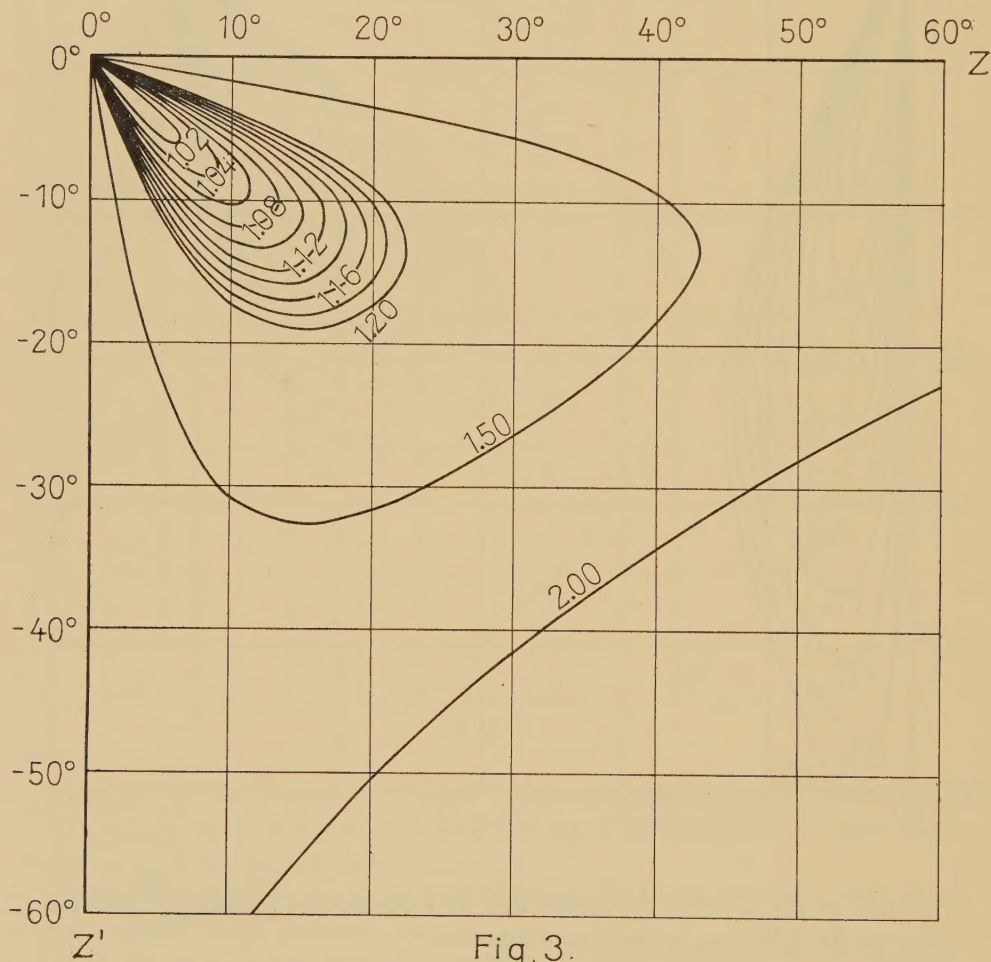


Fig. 3.

If two longitude stations have the same latitude, the greatest possible weight can be obtained only in one way—that is, by choosing all the stars in zenith. But this solution is unstable and must therefore be rejected. Even the smallest deviation from zenith may produce a very great loss of weight.

For this reason we always have to observe one group of stars south of zenith and another group north of zenith. If we choose a small zenith distance for the two groups, we get a great weight but only a small stability, so that a missed star will cause a considerable loss of weight. We therefore have to make a compromise and choose the zenith distance sufficiently great to

secure a good stability combined with a reasonably great weight of the clock correction. The curves show that this can be done in several ways, and that it is possible to obtain a sufficient stability together with a weight which for low or medium latitudes is only 5 per cent. below the maximum value, while for high latitudes we must acquiesce in a somewhat heavier loss in weight. If we take the two groups equally great, the northern group should be chosen a little closer to zenith than the southern one. If we choose one group greater than the other, the greater group should be advanced towards zenith and the smaller one removed from zenith in the proportion indicated by the figures.

At low latitudes it is advantageous to choose ν equal to 1 or in the neighbourhood of 1. No polar stars are then observed. For instance, for $\phi = 30^\circ$, $\rho = 1$, we may choose a star of the declination 44° together with a star of 14° declination, but we may just as well combine a star of 50° declination with two stars of 19° declination, or 1 star of 61° declination with 5 stars of an average declination of 24° .

At medium latitudes it is reasonable to choose ν somewhat greater than 1, because it is difficult to find a sufficient number of bright stars north of zenith. Yet ν should never be chosen greater than, say, 5, as a polar star may be missed owing to clouds or during the registration of a time-signal. For $\phi = 45^\circ$, $\rho = 1$, we get a very good weight ($g = 1.05$) if we combine a polar star of 71° declination (upper culmination) with 5 clock stars having an average declination of 40° , and it is of no advantage to choose the polar star very close to the pole. The same relative weight may be obtained by combining a star of 62° declination with two stars having an average declination of 36° , or a star of 57° declination with a star of 31° declination.

At high latitudes stars with rather great declinations should be preferred, and all available polar stars should be observed. If, for instance, $\phi = 75^\circ$, $\rho = 1$, it is advantageous to choose $\nu = 2$, and in this case a good weight of the clock correction ($g < 1.5$) is obtained by choosing a close polar star ($\delta' = 90^\circ$) combined with two stars south of zenith with an average declination $\geq 52^\circ$; or a star of 80° declination in the lower culmination together with two stars south of zenith having an average declination $\geq 49^\circ$; or a star of 65° declination in the lower culmination together with two stars south of zenith having an average declination $\geq 56^\circ$.

Again, we obtain $g < 1.5$ by combining a close polar star ($\delta' = 90^\circ$) with a star in the upper culmination, having the declination greater than 33° and smaller than 72° . Several other combinations also give a good weight. It will be seen that by means of a small number of weight curves we can easily choose one or more groups of stars suitable for a given latitude.

If the same stars are observed on two longitude stations with different latitudes the obtainable weight will be somewhat smaller. In this case preference should be given to the northern station. A glance at the figures shows at once how to choose the stars.

I now turn to the azimuth. From (3) the mean error μ_k in the azimuth k is found to be

$$\mu_k^2 = \frac{m_t^2 + m_{t'}^2}{(K - K')^2}, \quad (10)$$

and as we have

$$K - K' = \frac{\sin(p - p') \cos \phi}{\sin p \sin p'},$$

we get

$$\mu_k^2 = \frac{b^2 \sec^2 \phi}{N} f\left(p, p', \nu, \frac{\pi}{2}\right). \quad (11)$$

It follows from this equation that all the statements made in the foregoing are also valid for the mean error of the azimuth if we replace the zenith distances z and z' by the polar distances p and p' , and if we take the function f which corresponds to the pole, $\phi = \frac{\pi}{2}$, where, however, it has nearly the same values as at high latitudes. In the actual case we have

$$f_2\left(p, p', \nu, \frac{\pi}{2}\right) = (1 + \nu) \left(1 + \frac{1}{\nu}\right) \frac{\sin^2 p \sin^2 p'}{\sin^2(p - p')}.$$

If astronomical longitude and azimuth determinations are carried out in some of the trigonometrical points of a geodetic net, these determinations may be used in the adjustment of the net to check the occurrence of systematic errors of direction, the effects of which might be summed up along the chain and cause a considerable error in the result. In the Laplace equation for geodetic and astronomically determined points, the astronomical longitude l and azimuth k enter in the combination $k - l \sin \phi$. As a rule k and l are determined by different measurements. But if it is required to observe the longitude and azimuth simultaneously by means of a meridian mark, the question arises how to choose the stars so that the quantity

$$L = k - l \sin \phi \quad (12)$$

may be found as exactly as possible. This problem has recently been treated by J. Schive,* and for $\rho = 0$ by Th. Niethammer.†

The equations (2) and (3) may be written as follows:—

$$u = \frac{l \sin z' \cos(z - \phi) - l' \sin z \cos(z' - \phi)}{\cos \phi \sin(z' - z)} - i \sec \phi,$$

$$k = \frac{(l - l') \cos(z - \phi) \cos(z' - \phi)}{\cos \phi \sin(z - z')} + i \tan \phi.$$

From this we derive

$$\left. \begin{aligned} L \cos \phi \sin(z - z') &= l \cos(z - \phi)(\cos z' \cos \phi + 2 \sin z' \sin \phi) \\ &\quad - l' \cos(z' - \phi)(\cos z \cos \phi + 2 \sin z \sin \phi) \\ &\quad + 2i \sin \phi \sin(z - z'). \end{aligned} \right\} \quad (13)$$

If we define an angle α by the equations

$$\begin{aligned} r \sin \alpha &= \cos \phi, \\ r \cos \alpha &= 2 \sin \phi, \end{aligned}$$

* *A.N.*, 259, 218, 1936.

† *A.N.*, 260, 225, 1936.

then

$$\tan \alpha = \frac{1}{2} \cot \phi, \quad (14)$$

$$\cos z \cos \phi + 2 \sin z \sin \phi = \sqrt{1 + 3 \sin^2 \phi} \sin(z + \alpha),$$

which inserted in (13) gives

$$L = \sqrt{1 + 3 \sin^2 \phi} \frac{l \sin p \sin(z' + \alpha) - l' \sin p' \sin(z + \alpha)}{\cos \phi \sin(z - z')} + 2i \tan \phi. \quad (15)$$

Neglecting the influence of the level error, we derive from this equation in connection with (4) the following expression for the mean error $\mu(L)$ in L :—

$$\mu^2(L) = \frac{b^2 \sec^2 \phi}{N} (1 + 3 \sin^2 \phi) f(z + \alpha, z' + \alpha, \nu, \phi + \alpha). \quad (16)$$

It will be seen that even this error depends on the function f discussed above. As $\phi + \alpha$ is a large angle, we must here consider the function f for arguments corresponding to high latitudes. For, from (14), we see that

$$\tan(\phi + \alpha) = \cot \phi + 2 \tan \phi.$$

Hence $\phi + \alpha$ attains a minimum value when

$$\sin \phi = \sqrt{\frac{1}{3}},$$

that is, when

$$\cos(\phi + \alpha) = \frac{1}{3}.$$

From this it follows that at all latitudes

$$90^\circ \geq \phi + \alpha > 70^\circ.$$

If by means of (8) we substitute g for the function f , (16) may be written as follows:—

$$\mu^2(L) = \frac{a^2 + b^2 \sec^2(\phi + \alpha)}{N} \sin^2 \phi g(z + \alpha, z' + \alpha, \nu, \phi + \alpha). \quad (17)$$

This formula invites us to employ stars of rather great declinations. For instance, if the latitude be 70° , we get a good weight of L by combining a polar star in lower culmination with one or two stars a little south of zenith. If the latitude is 45° , we have $\alpha = 27^\circ$, and for $\rho = 1$ we shall obtain a value of g less than 1.2 if, for instance, we combine a polar star in upper culmination with a star with a declination of between 41° and 66° , or a polar star with two stars with declinations between 57° and 69° .

The solution recommended by J. Schive consists in using $N - 1$ stars, all of the zenith distance $z = -\alpha$, and a single star at an angular distance of 45° from the others. In this case we have $g = 1 + \frac{1}{\nu}$, and (15) is reduced to

$$L = -l \sin \phi + 2i \tan \phi,$$

L being independent of l' . For every combination of stars corresponding to points situated in the domain bounded by the curve $g = 1 + \frac{1}{\nu}$ we have

$$1 < g < 1 + \frac{1}{\nu}.$$

The influence of the level error i on the longitude or the azimuth is independent of the selection of the stars. If m_i is the mean error in a single determination of the inclination, it follows from (2) that the contribution of the level error towards the square of the mean error of the clock correction is

$$\frac{m_i^2}{N} \sec^2 \phi.$$

For the azimuth we find in a similar way from (3),

$$\frac{m_i^2}{N} \tan^2 \phi.$$

With careful observations m_i can be estimated to $\pm 0^s.008$.

For precise longitude determinations it is of the greatest importance to have a good right ascension system. If different stars are observed in the two longitude stations, the influence of the right ascension errors may become considerable; but even when the same stars are employed at both places, all the stars will be observed at different zenith distances on the two stations, unless the difference of latitude is very small. Right ascension errors which depend on the declination will cause an azimuth error which is different at the two stations.

Let us first assume that the latitudes ϕ_1 and ϕ_2 of two longitude stations are not very different and that at both stations the same stars are employed, namely, n' stars of the right ascensions α_i' and the pole distances p' , and n stars of the right ascensions α_i and the pole distances p . We will consider the terms in the expression for the longitude difference which depend on the right ascensions. These are:

$$\frac{\frac{1}{n}[\alpha_i] \sin p \cos (\phi_1 + p') - \frac{1}{n'}[\alpha_i'] \sin p' \cos (\phi_1 + p)}{\cos \phi_1 \sin (p - p')} \\ - \frac{\frac{1}{n}[\alpha_i] \sin p \cos (\phi_2 + p') - \frac{1}{n'}[\alpha_i'] \sin p' \cos (\phi_2 + p)}{\cos \phi_2 \sin (p - p')}.$$

This expression can be reduced to

$$\left(\frac{1}{n'}[\alpha_i'] - \frac{1}{n}[\alpha_i] \right) \frac{\tan \phi_1 - \tan \phi_2}{\cot p' - \cot p}. \quad (18)$$

Suppose the accidental mean error m_a in the right ascension to be of the form

$$m_a^2 = c_1^2 + c^2 \sec^2 \delta, \quad (19)$$

where c and c_1 are constants for which we put $c_1 = c\rho$. From (18) we find for that part $\mu_a(\lambda)$ of the mean error in the longitude difference λ which results from the errors in the right ascensions,

$$\mu_a(\lambda) = \pm \frac{c}{\sqrt{N}} (\tan \phi_2 - \tan \phi_1) \sqrt{f(p, p', \nu, \frac{\pi}{2})}. \quad (20)$$

Here $N = n + n'$ is the total number of stars in the set and f the function discussed above. From the curves showing the function f we can read the magnitude of this error without undertaking any new computation.

If we put

$$\mu_a^2(\lambda) = m'^2 + m^2,$$

where m' and m are the parts of the mean error originating from the stars of the pole distances p' and p respectively, we find from (20)

$$\frac{m}{m'} = \pm \frac{1}{\sqrt{\nu}} \frac{\sin p'}{\sin p} \sqrt{\frac{1 + \rho^2 \sin^2 p}{1 + \rho^2 \sin^2 p'}}. \quad (21)$$

If corresponding to $n' = 1$ we observe one polar star of the pole distance p' and ν clock stars of the mean pole distance p , it follows from this equation that m is much smaller than m' because p' is small. The errors in the right ascensions of the clock stars are thus almost without influence on the longitude difference.

From (20) we further obtain

$$m' = \pm c (\tan \phi_2 - \tan \phi_1) \frac{\sin p}{\sin (p - p')} \sqrt{1 + \rho^2 \sin^2 p'}. \quad (22)$$

If $p' = 0$, m' is independent of p . If p' is small, the fraction $\frac{\sin p}{\sin (p - p')}$ varies slowly with p , and it will be smallest for negative values of p' . It is therefore advantageous to take the polar star in its lower culmination, and the error in the longitude difference depends only to a small degree on the pole distances of the clock stars except in the case of high latitudes.

If in (21) we put $\nu = 1$ and observe two groups of zenith stars, we find that m and m' are of about the same order of magnitude, with some preponderance for the error originating from the northern group.

That part of the mean error in a longitude difference which results from the transit observations is nearly proportional to $\sec \phi_1$. However, the factor $\tan \phi_2 - \tan \phi_1$ contained in (20) is roughly proportional to $\sec^2 \phi_1$ if $\phi_2 - \phi_1$ is kept constant, because we have

$$\tan \phi_2 - \tan \phi_1 = \frac{\sin (\phi_2 - \phi_1)}{\cos \phi_1 \cos \phi_2}.$$

The right ascension errors are therefore of greater importance at high than at low latitudes. To some extent, however, the increase of $\tan \phi_2 - \tan \phi_1$ with the latitude is compensated by the last factor of (20). At small or medium latitudes the function $f\left(p, p', \nu, \frac{\pi}{2}\right)$ will assume rather high values if the stars are chosen as indicated above—that is, in such a way that the error in the clock correction originating from the transit observations becomes small. Equation (20) shows that $\mu_a(\lambda)$ approaches its minimum value when we choose the stars that give the best determination of the azimuth. At high latitudes the two principles of selection agree rather well, because the pole distance and

the zenith distance do not differ much, and in the selection of stars a compromise can easily be made which approximately minimises the error originating from the transit observations as well as that originating from the right ascension errors. But at lower latitudes there is a conflict. The stars which are favourable for the transit observations give (20) a magnitude that is far from the minimum value and may be 4 times as great. Conversely, if we demand that (20) shall be in the neighbourhood of its minimum, the error in the longitude determination originating from the transit observations is considerably increased.

The question now arises whether we should try to compromise between the two different principles for selecting the stars. The answer depends on the magnitude of the product $c (\tan \phi_2 - \tan \phi_1)$ —that is, on the precision of the right ascension system and on the latitude difference.

Nowadays it is not difficult to secure a good right ascension system. If necessary, we may derive corrections for the adopted right ascensions from the longitude determinations and carry out a readjustment with the improved right ascensions. In the longitude determinations made by the Baltic Geodetic Commission in 1929, the right ascensions of the stars employed were determined by Renz at the observatory at Pulkovo previous to the longitude measurement. The measurements showed that the mean error of the right ascensions of the clock stars in Renz's catalogue was $\pm 0^s.006$. This corresponds to a value of the constant c which does not exceed $0^s.004$. With such a small value of the quantity c the influence of the accidental right ascension errors on the longitude difference is imperceptible, even if the latitude difference rises to, say, 10° , and we may quite ignore the error which is expressed by the equation (20).

If the latitude difference or the longitude difference between the two stations is great, it becomes preferable to observe different stars at each station. If z_1 and z_2 denote the meridian zenith distances, and ϕ_1 and ϕ_2 the latitudes of the two stations, we now find the following expression for $\mu_a(\lambda)$:—

$$\mu_a^2(\lambda) = \frac{c^2 \sec^2 \phi_1}{N} f(z_1, z_1', \nu, \phi_1) + \frac{c^2 \sec^2 \phi_2}{N} f(z_2, z_2', \nu, \phi_2). \quad (23)$$

By comparing this expression with (6) it will be seen that in this case the same stars will reduce both μ_a and μ_u to values in the neighbourhood of their minima. It will then be natural to choose the stars independently for each of the two stations, so that a good determination of the clock correction is obtained in each place. A comparison between (20) and (23) leads to a criterion whereby we can judge at which value of the difference of latitude $\phi_2 - \phi_1$ it becomes preferable to employ different stars at the two longitude stations. We find the following approximate condition:—

$$f\left(p, p', \nu, \frac{\pi}{2}\right) \sin^2(\phi_2 - \phi_1) < (\cos^2 \phi_1 + \cos^2 \phi_2) \left(1 + \rho^2 \cos^2 \frac{\phi_1 + \phi_2}{2}\right).$$

This inequality shows that it may be advantageous to observe the same stars at both stations, even if the latitude difference is not very small, but in considering this we must of course not forget that the error in the clock correction

must not move too far from its minimum. The final decision then depends on the magnitude of the constant c . In certain longitude determinations this quantity has assumed rather high values. If, however, c is small, the error determined by (23) also becomes small provided the stars are suitably chosen, and in this case the influence of the accidental right ascension errors on the longitude determination is negligible.

It is otherwise with the systematic errors of the right ascensions, which are always dangerous. Expression (18) shows that if we always use the same method in selecting the stars, these errors will have a very constant and quite appreciable influence on the longitude. The systematic errors follow no simple law and therefore cannot be made the subject of mathematical treatment. There are, however, two means of partly eliminating the influence of the systematic errors. We have seen that there are several equally good ways of approximately minimising the mean error μ_u that originates from the transit observation. We may either give the parameter ν a high value and observe one polar star together with 5 or 6 clock stars a little south of zenith; or we may put $\nu = 1$ and observe two equal groups of stars north and south of zenith. Or again, we may give ν an intermediate value and observe one group of stars between the pole and zenith and another somewhat larger group south of zenith. The expression (18) shows that by shifting between the various possible values for ν , which are equally good, we obtain a variation of and a partial elimination of the influence of the systematic right ascension errors. Furthermore, it is of advantage to alternate between polar stars in the upper and lower culmination, as the predominant term $\cot p'$ of the denominator changes its sign with p' . It is easy to see that these statements are also valid in case different stars are observed at the two stations.

The final conclusion is that the rather limited liberty allowed us in the selection of stars should be utilised to its full extent. And care should be taken not to give absolute preference to one single way of selection, such as, for instance, always making use of the polar-star method as many observers do. It is preferable to give equal consideration to each of the few equally good solutions which present themselves.

I now turn to another point. When we use a moving wire micrometer the personal equation of the observer is much smaller than by the older methods, but it is not negligible. For a slowly moving star the personal equation is not the same as for the rapidly moving equatorial stars. This is another reason for choosing the circumpolar stars not too close to the pole and for interchanging between different values of the parameter ν .

When determining longitudes for geodetic purposes I have found it profitable to make use of the following method: During three nights three observers carry out measurements simultaneously at their different stations. Then a cyclic permutation of the observers takes place, and the measurements are continued through a second period of three nights. By this triangular method the personal equation is completely eliminated, and the work involved is considerably smaller than in the case of a complete interchange of observers. I prefer short series of three nights to long series, because the personal equation and the seasonal error do not remain constant, so that they

are best eliminated by a rapid interchange. If we wish to increase the weight, it will be better to repeat the measurements at another time than to extend the observations over many nights between each interchange. The method I have sketched has been employed on a somewhat larger scale by the Baltic Geodetic Commission for the determination of the longitude differences between the central stations of eight countries round the Baltic.

In the longitude determinations carried out in Denmark, I have succeeded in eliminating the errors resulting from variations of the clock rate by transferring the seconds of the principal clock in Copenhagen to the field stations by means of our own wireless transmitter. The signals are received at the field stations and registered by means of undulators, in which the same stylus records the wireless signals and the impulses of the transit observations. The corresponding kicks are in opposite directions, so that they are easily identified. No relays are included in the electrical circuits.

Mr. J. Rybner * of the Danish Geodetic Institute has called attention to the danger of systematic errors introduced by relays or ordinary chronographs. The time of reaction of these devices ranges from 5 to 30 and from 20 to 50 milliseconds respectively, and it varies widely according to the working conditions. If the tension of the battery diminishes, the time of reaction increases considerably, even if the resistance of the circuit is adjusted to keep the current constant. The settings of the different adjustment screws also influence the time of reaction. The difference between the rapidity obtained by the direct-current micrometer impulse and by the rectified wireless signal is less important, as it is eliminated by the interchange of observers, provided it remains constant. The uncertainty of the time of reaction is much reduced if light instruments working with small electric currents are used. Especially suitable is the undulator built by the Great Northern Telegraph Company. This instrument, which has been used by the Danish Geodetic Institute since 1923, has a recording stylus consisting of a thin silver tube, attached to two steel magnets revolving in a field formed by the line current. It gives a good kick for one milliamperere at one volt, and it can therefore be connected direct to the rectifier stage of the wireless receiver without any relay. Its time of reaction is about seven milliseconds and is very constant, as only two milliseconds are due to the delayed growth of the operating current. The other five milliseconds are due to the inertia of the stylus, which causes it to be slightly bent at the beginning of its motion, and this part of the operating time is quite constant under all working conditions. It is not even influenced by the friction of the pen on the paper strip.

If λ and k are the astronomical longitude and azimuth, while λ' and k' are the geodetic longitude and azimuth, we have the Laplace equation

$$k - k' - (\lambda - \lambda') \sin \phi = 0.$$

This equation gives, as is well known, a valuable control, and it can be employed with advantage in the adjustment of a geodetic net, as it serves to eliminate the systematic errors which might accumulate along a chain of great length.

* *Verh. 5 Tag. d. Balt. Geod. Kom.*, 234, 1930; *Verh. 8 Tag. d. Balt. Geod. Kom.*, 162, 1935.

The question now arises what the distance between the Laplace points of a given triangulation should be. Opinions differ very much. In many countries the Laplace points are chosen very far from each other, so that the astronomical observations obtain a high relative weight in comparison with the geodetic measurements. On the other hand, in some of the recent triangulations a very great number of Laplace points are used. For the determination of the deflections of the vertical we need only the latitude and azimuth or the latitude and longitude. The inclusion of both the longitude and the azimuth not only requires an increased work of observation, but also involves much more extensive calculations of adjustment. On the other hand, if a too small number of Laplace points is observed a considerable loss of accuracy is suffered. The choice of Laplace points is, therefore, a matter of great importance.

I had to face this problem in two concrete cases, namely, the triangulations carried out in later years in Denmark and Greenland under my personal leadership. In Denmark, quite a new triangulation covering most of the country has been measured with great care. The net is connected with six bases, namely, Sorring, Tarm, Brønderslev, Döstrup, Saksköbing and Amager, and the longitudes are measured at 27 stations. In Greenland a long first-order chain has been extended along the west coast near the edge of the inland ice which covers the interior. The chain is finished between latitudes 61° and 75° and will be carried further towards the north. The longitude and azimuth were measured at Julianehaab, Ivigtut, Kornok, Green Ejland, Kekertat and Satut-Satorssuaq.

In order to decide how many Laplace points should be observed we must have a criterion for what may be considered necessary. Such a criterion can be derived from the error ellipse determining the mean error of a trigonometric point. If the eccentricity of the error ellipse is great, the mean error will be considerably larger in the direction of the major axis than in the directions perpendicular to it, and the triangulation cannot be considered well balanced. Accordingly, the criterion must be that so many Laplace points are selected that the error ellipses become approximately circles.

For the new Danish triangulation I have computed the error ellipses and found that these are nearly circles if the distances between the Laplace points do not exceed 250 km. For Greenland the corresponding figure is 230 km. Actually I had expected a somewhat greater difference between these two numbers, but apparently the great difference of latitudes is not of much moment. If the distances between the Laplace points are considerably greater than 230 km. it is not taking full advantage of the precision of the angle measurement. On the other hand, if the Laplace points are considerably closer, we have to do more astronomical work than is needed and the adjustment becomes unnecessarily complicated.

I just mentioned that we have recently carried out a number of longitude determinations in Greenland, and I shall conclude with a short remark on these observations. We know that Wegener considered that his hypothesis of the displacement of continents had been supported by longitude determinations in Greenland, and from them he calculated that Greenland is

moving westwards about 20 metres a year. But the old observations used by Wegener were carried out with primitive instruments and he overrated their exactitude. In 1927 and 1936 the Danish Geodetic Institute carried out longitude determinations at Kornok on the west coast of Greenland with a first-class transit instrument, both times on the same pillar. Practically speaking the two measurements gave the same result. It is most likely that the deviations of the old observations from the new ones are the result of observation errors.

RED SHIFTS AND THE DISTRIBUTION OF NEBULÆ.

Edwin Hubble.

1. In *M.N.*, 97, 1937 January, appear two rediscussions, by Sir Arthur Eddington and by G. C. McVittie, of results of an analysis of nebular distribution recently published by the writer.* Five surveys to apparent magnitudes $m = 18.47$ to 21.03 furnished a relation $N_m = f(m, \Delta m)$, where N_m is the number of nebulæ per square degree brighter than m , and Δm is the average correction for the dimming effects of red shifts on nebulæ of magnitude m . The true, or corrected, distribution was found to be so nearly uniform, on any interpretation of red shifts, that the investigation could be reversed. Assuming true uniformity and referring apparent departures to effects of red shifts, an attempt was made to distinguish between the alternative, possible interpretations of red shifts as Doppler effects or not Doppler effects.

The problem may be stated in another way. A rapidly receding object will appear fainter than a stationary object at the same momentary distance. Consequently, apparent luminosities of nebulæ offer different scales of distance depending upon whether or not the recession factor is involved. It seemed possible that, when the laws of red shifts and of nebular distribution were formulated on the alternative scales, the wrong scale might introduce anomalies which could be detected or at least suspected. The possibility was partially realized, for apparent anomalies were found in the scale involving recession factors, which could be eliminated only by a forced interpretation of the data.

2. The numerical results of the investigation were as follows: The law of nebular distribution, regardless of interpretation of red shifts, is well represented by the relation

$$\log_{10} N_m = 0.6(m - \Delta m) - 9.052 \pm 0.013, \quad (1)$$

where †

$$\log_{10} \Delta m = 0.2(m - \Delta m) - 4.239 \pm 0.024. \quad (2)$$

* "Effects of Red Shifts on the Distribution of Nebulæ," *Mt. Wilson Contr.*, No. 557; *Ap. J.*, 84, 517, 1937.

† The probable error in (2) was incorrectly published as ± 0.008 , which refers to a single regression curve, assuming a particular relation.

If we represent Δm by $B \cdot d\lambda/\lambda$, then B is about 3.0 if red shifts are not velocity-shifts, or about 4.0 if red shifts are velocity-shifts.

Case I. Assume that red shifts are not velocity-shifts. The law of red shifts, for isolated nebulæ, is then *

$$\log_{10} d\lambda/\lambda = 0.2(m - \Delta m) - 4.707 \pm 0.016. \quad (3)$$

Combining (2) and (3),

$$\log_{10} \Delta m = \log_{10} d\lambda/\lambda + 0.468 \pm 0.029 \quad (4)$$

and

$$\Delta m = (2.94 \pm 0.22)d\lambda/\lambda. \quad (5)$$

Thus the assumption leads to a consistent picture of the observable region. The law of red shifts is linear, and the distribution is uniform over the complete range of the observations. There is no evidence of spatial curvature. The sample is homogeneous, and the universe presumably extends indefinitely beyond the limits of our telescopes.

Case II. Assume that red shifts are velocity-shifts ($B=4$). The expected N, m relation, derived from the theory of expanding universes which obey the relativistic laws of gravitation, is †

$$\log_{10} N_m = 0.6(m - \Delta m) + F + \text{const.}, \quad (6)$$

where F is a term involving spatial curvature. The empirical law of red shifts may be written

$$d\lambda/\lambda = 5.37 \times 10^{-10}r + 2.54 \times 10^{-19}r^2, \quad (7)$$

where r is expressed in light years.

Since B is already given, and r is known in terms of $(m - \Delta m)$, these equations furnish a value of R_0 , the present radius of curvature. Moreover, since the coefficients in (7) represent the first- and second-time derivatives of R_0 , it is possible to determine the parameters of a relativistic expanding universe that is consistent with the observations. The model is closed, small and dense—"an ever-expanding universe of the first kind" in which

$$k = +1, \quad R_0 = 4.7 \times 10^8 \text{ lt. yr.}, \quad \rho = \text{about } 6 \times 10^{-27} \text{ gm./cm.}^3$$

and

$$\Lambda = +(4.4 \text{ to } 6.9) \times 10^{-18} \text{ yr.}^{-2}.$$

3. Eddington makes the criticism that, in deriving equations (1) to (5), the possible effects of dispersion in the absolute magnitudes of nebulae

* *Mt. Wilson Contr.*, No. 549; *Ap. J.*, **84**, 270, 1936. The probable error in equation (3) was not given, but it is readily calculated from the data listed in Table II.

† "Two Methods of Investigating the Nature of the Nebular Red Shifts," Hubble and Tolman, *Mt. Wilson Contr.*, No. 527; *Ap. J.*, **82**, 302, 1935. This paper adapted the general formulæ previously developed by Tolman for application to the observational data. A preliminary statement of the present status of the observational work was included, which was developed in detail in the later contribution now under discussion.

have been neglected. He then investigates the effects of dispersion by a very neat, original method, and finds that the necessary correction, $B = 3.06$ instead of 2.94, "proves to be unimportant." This independent confirmation of the analysis of the data is most welcome.

However, it may be pointed out that effects of dispersion had already been examined. A rigorous expression for N_m , taking full account of dispersion in M , was developed in the above-mentioned discussion of methods and principles by R. C. Tolman and the writer.* Actual solutions of the equation by numerical integration indicated "that on account of the fairly sharp maximum exhibited by the exponential factor $e^{-\frac{(M - \bar{M})^2}{2\sigma^2}}$, the integral . . . can often be satisfactorily approximated by assigning to the rest of the integrand the value that it has at the maximum point where $M = \bar{M}$." In other words, the rather complicated, rigorous method could be replaced by the very simple method actually used, without introducing significant errors. It is this statement that has been checked by Eddington's independent investigation.

4. His second and more serious criticism is that, although the value of B indicated by the data is approximately correct, it is not sufficiently reliable to constitute a discordance with the expected value 4.0. He states that the discordance (3.06 against 4.0) is equivalent to a discordance † of 0.066 mag. in the range of the measured m . For, he adds, let δm be this range (2.56 mag.). From δm and the assumption of uniform distribution we derive $\delta(m - \Delta m)$ and, consequently, $\delta(\Delta m)$, from which B is determined. "An error of 0.01 mag. in δm will produce an equal error in $\delta(\Delta m)$ and an error of 0.13 in k " (where k , in his notation, is B in the present discussion).

But, by differentiating (2),

$$d(\Delta m) = \frac{\Delta m}{2.171 + \Delta m} \cdot dm, \quad (8)$$

and an error in m evidently produces a much smaller error in Δm . Errors in δm , of course, produce comparable errors in $\delta(m - \Delta m)$, which are reflected in the constants of equations (1) and (2), and, ultimately, in B . The calculation is delicate, and the probable error in (5), of the order of 1/5 the discordance, does not inspire complete confidence. To this extent the criticism is justified.

The uncertainty arises from the fact that, over the short range δm , the constants can be adjusted with considerable latitude. A material increase in δm improves the situation, and, as it happens, the only other published survey, the Harvard catalogue, complete to $m = 12.8$, and with $\log \bar{N} = -1.38$, meets this requirement. When these data are included in the least-squares solution, the constants in (1) and (2) are essentially unchanged but the weights are considerably increased. The coefficient is

* *Mt. Wilson Contr.*, No. 527; *Ap. J.*, **82**, 302, 1935.

† By the writer's method of analysis, least-squares solution of the data in all the surveys, an error in δm of the order of 0.12 mag., evenly divided between the two extreme limits, is required to furnish $B = 4.0$.

then $B = 2.92 \pm 0.16$ which, in a formal sense, is significant. Because of the small numbers of nebulæ involved, the propriety of including the Harvard list on the same basis as the deeper surveys may be questioned. They are mentioned merely to emphasize the point that the only additional information now available tends to improve the reliability of the original results.

However, the measure of confidence to be placed in equation (5) does not depend primarily on the probable errors. The major uncertainties must be sought in the possibilities of systematic errors, and there is found the real significance of the conclusions. When it became evident that B would fall below the expected value, 4.0, the observations were reduced by methods which, at each step, favoured a large value. The methods were described, and it was specifically mentioned that the "examination of possible sources of systematic errors strongly suggest that the data . . . exhibit a maximum range in m and a minimum range in $\log \bar{N}$. The conclusion has an important bearing on the interpretation of the data."

As an example, consider the determination of δm . A zero-point was established at 19.11 ± 0.02 on the scale of the polar sequence, for standard exposures of 20 min. with the 100-inch, from measures of *schraffierkassette* images. The range, δm , is not very sensitive to errors in the zero-point, and precision was sought primarily for the purpose of comparing (2) and (3). The brightest limit, 18.47 ± 0.03 , for 20 min. exposures with the 60-inch, was determined from the same plates that furnished the zero-point, and, consequently, the uncertainty in that section of the range, aside from the small accidental errors, is the presumably negligible scale-error of the polar sequence over the short interval of 0.64 mag. The faintest limit, 21.03 ± 0.03 , for 120 min. exposures with the 100-inch, corresponds with an extrapolation from the standard 20 min. exposures, using a reciprocity law with an exponent of 0.99.

This section of the range, 1.92 mag., is clearly more unreliable than the short, brighter section, and the uncertainties operate in one direction only, namely, to reduce the interval. In order to appreciably increase δm , it would be necessary to postulate an exponent greater than unity in the reciprocity law; in other words, to assume that, when the 20 min. exposures are lengthened six times, nebulæ are recorded that are *more than* six times fainter. Under the particular conditions, faint objects and Eastman 40 plates, the assumption would be wholly unreasonable. The adopted range is clearly a maximum, and any significant revision would be expected to reduce the interval. The other observable quantity, N_m , and the calculation of the expected values of B , were treated from the same point of view.

The interpretation of red shifts by the theory of expanding universes is so plausible and so widely current that, in making a delicate test of the theory, it is desirable to push uncertainties in the favourable direction before admitting a discordance. Nevertheless—and this is perhaps the significant result of the investigation—when the observational data are weighted in favour of the theory as heavily as can reasonably be allowed, they still fall short of expectations.

With the theory in its present form, and with the observations now available, the discordance exists when spatial curvature is neglected.*

5. McVittie is concerned only with the determination of parameters in Case II. He attempts to show that the observational data are consistent with an infinite universe with negative curvature rather than a finite universe with positive curvature. The point of departure is the well-known pair of equations,

$$\kappa\rho = -\Lambda + 3(\dot{R}/R)^2 + 3c^2k/R^2, \quad (9)$$

$$\kappa\dot{p} = \Lambda - (\dot{R}/R)^2 - 2\ddot{R}/R - c^2k/R^2. \quad (10)$$

Making the usual assumption that the pressure p is negligible, and eliminating Λ , the result is

$$2c^2k/R_0^2 = \kappa\rho - 2(\dot{R}_0/R_0)^2 + 2\ddot{R}_0/R_0, \quad (11)$$

where the subscript zero is used to indicate the present epoch.

Now the derivatives, $\dot{R}_0/R_0$ and $\ddot{R}_0/R_0$, are determined, at least as far as their signs and orders of magnitude are concerned, from the observational data. The writer, following Tolman,† calculates them (or their equivalents) directly from the coefficients in the law of red shifts represented by (7). McVittie adopts an approximate value of $\dot{R}_0/R_0$ corresponding with the first-order term in (7), but derives $\ddot{R}_0/R_0$ by another, rather arbitrary method that will be discussed presently. However, the two sets of values are of the same general order. Thus

	$\dot{R}_0/R_0$	$\ddot{R}_0/R_0$
Hubble	1.70×10^{-17}	-2.84×10^{-34} .
McVittie	2×10^{-17}	-7×10^{-34} .

Using the first set, and remembering that $\kappa = 1.674 \times 10^{-6}$, equation (11) becomes

$$1.075 \times 10^{27} k/R_0^2 = \rho - 6.85 \times 10^{-28}. \quad (12)$$

Thus k , which can be $+1$, 0 or -1 , depends upon whether or not the density ρ is greater than, equal to or less than the critical value 6.85×10^{-28} . If ρ is sensibly less than the critical value, k will be negative and R_0 will be of the order of 10^9 light years, as McVittie concludes. On the other hand, if the curvature is positive, ρ must be greater than the critical value, and,

* The statement is believed to be properly worded. The observations, of course, may involve errors sufficient to vitiate the conclusions, but, if so, the trouble will probably be found in hidden systematic errors. Further investigations of two kinds are desirable. Extensive surveys to bright limits, where the dimming effects are negligible, will tie down the constant in equation (1) and, consequently, restrict the possible adjustments of the constant in (2). Surveys to still deeper limits (which will be possible with the 200-inch reflector), where the dimming effects reach a magnitude and more, should then establish the value of B with a significance that is beyond controversy.

† The writer's estimate of parameters are essentially a revision, suggested by new data, of the earlier determinations made by Tolman, *Relativity, Thermodynamics and Cosmology*, p. 474, 1934.

as will appear later, greater than can be accounted for without introducing large quantities of internebular material. This latter situation is one of the anomalies which results from analysing the observational data on the basis of a distance scale involving recession factors.

McVittie approaches the problem by calculating ρ from the luminous nebulæ alone, determining α , the number of nebulæ per unit volume, and using Smith's value of the mean mass, $2 \times 10^{11} \odot$. By an oversight he has compared coefficients in two equations, one referring to nebulæ per square degree, and the other to nebulæ over the entire sky. Consequently, his result must be increased by a factor of the order of 4.1×10^4 . Moreover, when the Sun's mass is taken as 2×10^{33} gm. rather than the round number 10^{33} , McVittie's value of the density (8.2×10^{-33}) must be revised to

$$\begin{aligned}\rho &= 8.2 \times 10^{-33} \times 8.2 \times 10^4 \\ &= 6.7 \times 10^{-28},\end{aligned}\tag{13}$$

which is so near the critical density that the particular methods of approximation employed in the calculation become all-important.

In the writer's opinion, the most reliable value of α , the number of nebulæ per unit volume, is derived from equation (1), together with the expression for distance in terms of $(m - \Delta m)$ and $\bar{M}$, the mean absolute magnitude of nebulæ of a given m . Using the most recent value, $\bar{M} = -15.15$, and Smith's value for the mean mass, the density is

$$\begin{aligned}\rho &= 2.42 \times 10^{-73} \times 4 \times 10^{44} \\ &= 10^{-28}.\end{aligned}\tag{14}$$

Since Smith's result is considered as an upper limit, the density appears to be well below the critical value, and the anomaly persists—the derived curvature is negative and the radius is of the order of 10^9 light years.

The result is considered as an anomaly because, if the curvature were negative, then, after the corrections Δm are applied, the apparent distribution-density should fade out with distance, whereas, actually, it increases with distance. It was for this reason that the writer adopted the positive curvature necessary to compensate the apparently increasing numbers of nebulæ (in other words, to compensate the recession factor, without which the distribution appears to be uniform), and then derived the corresponding density (about 60 times the critical density).

6. The investigator must necessarily follow this course or else reject the apparently increasing α as errors in the observations or their interpretation. It seems highly improbable that the mean mass of nebulæ can be increased, or the mean absolute magnitude reduced, by amounts sufficient to remove the discrepancy. Consequently it is necessary, in order to preserve the theory intact, either to introduce large quantities of internebular material, or to question the validity of the corrections Δm as calculated from the relation, $\Delta m = 4d\lambda/\lambda$.

McVittie follows the latter course, and analyses the data by a method which seems to evade the difficulty at the cost of a rather arbitrary procedure.

He derived an N, δ relation (δ in his notation is the red shift $d\lambda/\lambda$) as follows:—

$$\log_{10} N = 0.501m - 7.371, \quad (15)$$

$$\log_{10} \delta = 0.2m - 4.967, \quad (16)$$

$$N = 1.113 \times 10^{5\delta^{5/2}} \quad (17)$$

and

$$dN/d\delta = 2.78 \times 10^{5\delta^{3/2}}. \quad (18)$$

Equation (15) is an interpolation formula, using the observed, uncorrected magnitudes, which satisfies the surveys well enough over the observed range in m , but which breaks down for brighter limits (for instance, the Harvard survey). Equation (16) represents an early formulation of the law of red shifts, in which the dispersion in M was not taken into account (the most recent value of the constant is about -4.707). Corrections Δm were applied but, since the distances were small, they may be neglected.

Equations (15) and (16) may be considered as linear interpolation formulæ, representing certain limited ranges of general non-linear relations. Near the observer the general relations are believed to be sensibly linear, the coefficients of m being 0.6 and 0.2 respectively. With increasing distance (increasing m) the slopes steadily diminish (due to the omission of the dimming effects, Δm). Consequently, the exponent of δ in (17), which is the ratio of the coefficients in (15) and (16), is arbitrary because it depends upon the particular ranges of the general functions which are selected for comparison.

The same ranges would presumably lead to an exponent, very nearly 3. But McVittie has compared a very remote region in (15) with a nearby region in (16), and necessarily derives a smaller exponent in (17) which happens to be almost precisely 5/2.

He wishes to compare coefficients in (18) with those in McCrea's theoretical expression for $dN/d\delta$. But the latter is a power series in δ , the first term of which is of the second order. Therefore, McVittie takes the further arbitrary step of expanding $\delta^{3/2}$ as a power series, also beginning with the second-order term. The step, in a sense, amounts to a rough approximation toward comparing the original functions in the same range. In that case the exponent, 2, would presumably appear as a matter of course, just as it appears when the Δm are included in the general relations. It is partly for this reason that the values of α and $\dot{R}_0/R_0$, which are derived from the coefficients of δ^2 and δ^3 , are of the same general order as those derived with the use of Δm , except for the oversight* in the case of α .

7. The advantages of an empirical N, δ relation have long been recognized, provided it extends out to large values of δ . Direct formulations, of course, are unpractical, and the comparisons of the N, m with the

* In equation (15) N refers to nebulæ per square degree, while in McCrea's formula N refers to nebulæ over the entire sky. Consequently, equation (17) should read $N = 4.59 \times 10^9 \delta^{5/2}$, and (18), $dN/d\delta = 1.148 \times 10^{10} \delta^{3/2}$, in order to make the comparison with McCrea's formula.

δ , m relation are significant only when the latter is extrapolated to the outer range of the former. At present, the most plausible extrapolation is that made with the use of the dimming factors, Δm . However unsatisfactory the method may be considered, it offers about the only means of estimating parameters now available.

For these reasons the direct evaluation of R_0 and its derivatives from the laws of distribution and of red shifts, separately, seems preferable to McVittie's indirect approach. Nevertheless, the observations as they stand lead to the anomaly of a closed universe, curiously small and dense, and, it may be added, suspiciously young. On the other hand, if red shifts are not primarily velocity-shifts, these characteristics vanish and the observable region appears as a small, homogeneous, but insignificant sample of a universe that may extend indefinitely both in space and in time.

Note.—Dr. McVittie has very kindly sent me a copy of a letter, to be published shortly in *Observatory*, in which he corrects the numerical errors in his original calculations. However, by adopting another, smaller value for the mean mass of nebulae, he preserves intact the significance of his original conclusions concerning spatial curvature and the cosmological constant.

The revisions do not affect the argument in the present discussion. In a *homogeneous, expanding* universe, a negative curvature would introduce an apparent thinning out of the nebular distribution (after corrections have been applied for the dimming effects of red shifts). Actually, we observe an apparently increasing distribution density. Unless the calculated dimming effects are seriously in error, a negative curvature can be adopted only at the expense of the postulated homogeneity or expansion.

Mount Wilson Observatory :
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STELLAR LUMINOSITY AND PHOTOSPHERIC PRESSURE.

E. A. Milne.

(Communicated by the Director of the University Observatory, Oxford)

1. *The General Problem.*—The data that may be supposed known about an actual star are its mass M , luminosity L , and radius r_1 . From these the *effective temperature* T_1 is calculated by the formula $L = \pi a c r_1^2 T_1^4$, where c is the velocity of light, a the radiation constant. Whether the colour-temperature or ionization temperature coincide with T_1 is here immaterial; a knowledge of L , M , r_1 is equivalent to a knowledge of L , M , T_1 .

The problem of stellar constitution may be then put in the following form: given L , M , T_1 or L , M , r_1 , what can be inferred about the constitution of the star in question? Can we infer anything about the equation of state of the material composing the star, or about its mean molecular weight, or about its opacity, or about the mode of transfer of energy in the interior, or about the distribution of density in the interior? The

star may be regarded as a sealed mechanism of which we can only view the exterior, but concerning which we know in addition certain global characteristics, namely L , M , and r_1 capable of measurement by an observer outside the star. Concerning the exterior, we view not merely an exterior surface, but an outer thickness of material, into which our observation penetrates until the onset of foggiess due to accumulating opacity cuts off further view of the deeper interior. We have in fact a fairly extensive knowledge of the stellar atmosphere and its zones, and some knowledge of the layers just behind the atmosphere which we call the photospheric layers. We know for example that these layers are gaseous, that their opacity contributes the star's continuous spectrum, and that the law of darkening of the solar disc towards the limb, together with the observed existence of limb-darkening in eclipsing binaries, indicates the presence of a temperature gradient which may be taken as approximately that for radiative equilibrium, *i.e.* that occurring when the transfer of energy through these layers to outer space takes place through radiation processes. The intensity and distribution of the continuous spectrum tells us something of the actual temperatures in these layers, ionization phenomena tell us something concerning jointly the pressures and temperatures (more precisely the electron-pressure), and the absolute and relative intensities of the lines in stellar absorption spectra have disclosed something of the ultimate chemical constitution and so of the mean molecular weight of the material in the reversing layer. As the reversing layer cannot be sharply distinguished from the outer photospheric layers, we thus know something of the mean molecular weight in the photospheric layers. From this and the temperatures and pressures the monochromatic opacity coefficients can be calculated from purely physical considerations, as has been done by Unsöld, Pannekoek and others, and from these, or more directly from the equilibrium, mechanical and radiative, of the photospheric layers, their general opacity may be derived.

All these facts and inferences from immediate observation we are entitled to take into our regard in considering the further problem of the deep interior. But when it comes to the interior, a different method has in the past been generally followed. This method is to set up some species of *model*, to compare the properties of the model with the observed global characteristics of the star and so to infer numerical values for some or all of the parameters occurring in the description of the model. It is then considered that these numerical values tell us something of the conditions in the actual stellar interior; for with the numerical values of the parameters evaluated, the temperatures, densities, pressures, etc., in the model become determinate. But from the pure theory-of-knowledge point of view the inference as to the actual conditions in the stellar interior is not legitimate unless the model is unique—that is, unless the model is the only one which reproduces the observed global characteristics. Otherwise, the arguments are not reversible—we cannot infer that the calculated state of affairs for the model applies to the actual stellar interior.

An example may make the matter clearer. A fundamental part has been played in the theory of stellar constitution by Eddington's "standard

model," in which (1) the material is taken as gaseous, (2) the molecular weight μ is taken as constant, (3) the energy-generation is taken as uniformly distributed, (4) the mechanism of transfer of energy is taken to be by radiation processes, (5) the opacity κ is taken as constant.* If we compare the properties of such a model with a star of observed L , M , r_1 , and assume a value for μ , we can derive a value for κ , as was originally done by Eddington. But we do not know to what part of an actual star (in which κ may be supposed to vary) the derived value of κ relates, or whether the derived value is a mean value for the actual star. This difficulty is not removed by adopting some *formula* for κ . For example, if we assume $\kappa = a\rho/T^{7/2}$, where a is a constant, then we may derive a value for a ; but if a for the actual star is not constant, we again do not know to what part of the star the derived value of a applies, or whether it gives a mean value, and so on. Again, if we assume a definite chemical composition for the model, *e.g.* assume a constant value for μ , and calculate κ as a function of μ , the same process will determine μ for the model. But we cannot be sure that this value of μ holds good for the actual star. The star may possess a gradient of chemical constitution, and we do not know to what part of the star (if any) the derived value of μ relates. This difficulty is again not removed by taking μ as variable according to some definite law, say $\mu = \mu_0(T/T_0)^t$, for again we shall not know to what part of the star (if any) μ_0 relates. For example, if the derived values of μ or μ_0 really hold good only for the exterior layers of the actual star, we have made no progress towards determining the interior composition of the star unless we can also determine the exponent t from observation. The star may possess a core of elements of high molecular weight which would be undisclosed if the derived value of μ related only to the photospheric layers.

These objections are not mere theoretical obstacles raised by an ultra-sceptical attitude. We shall see shortly that when we avoid the introduction of a specific model, essentially the same mathematical relations between the parameters emerge as for the standard model, but the parameters β , κ , and μ of the standard-model theory are shown to relate not to the stellar interior but to the photospheric layers. The derived values are not averages for the stellar interior, but apply solely to the outer layers. The reason for this is that whilst Eddington's model is essentially that of a polytrope of index $n=3$, the place where the model makes contact with observation, namely, the outer layers, is alone relevant to the calculation of luminosity, and the resulting relations depend only on n being equal to 3 in the vicinity of the outer layers. Substantially the same relations emerge if the compressibility in the outer layers corresponds to $n=3$, but corresponds to different values of n elsewhere in the star. Indeed a wide variety of hypotheses concerning the variations of κ , μ , and ϵ (the rate of generation of energy) through the star can be fitted together to give $n=3$ everywhere, but comparison with observation yields in all cases just the photospheric values of κ or μ .

* There is also the more general model " $\kappa\eta = \text{const.}$ " As $\eta=1$ at the boundary, and is known nowhere else, the computed value of " $\kappa\eta$ " determines the *boundary* value of κ (in agreement with the conclusions of the present paper) on this model.

In order to establish the results just stated, some new method of analysis of stellar structure must be attempted. This is the subject of the present paper. It is clear that to find out to what part of an observed star a derived value of a parameter relates, we must avoid using an *analytical formula* for the variation of the parameter, for the reasons stated; we must leave the variable in question as some quite undetermined function of position in the star, and then find out for what part of the star, or for what average through a determined portion of the star, the value of the variable is relevant to the observable characteristics. That is to say, we must proceed without adopting some specific model.

In that case, for the interior the only equation we have to rely on is the equation of mechanical equilibrium. For the outer or photospheric layers, we are entitled to use all the information that is available. When we carry out this programme, by fitting outer layers of observable characteristics on to an unknown interior in mechanical equilibrium, we find that we can calculate the luminosity of a star just as readily as when we make more extensive hypotheses. The calculation is that of the rate at which the outer layers of a compressible mass of material in mechanical equilibrium are losing energy to space. This leakage is made up from the interior, but the details of the way it is made up are irrelevant. Such details will affect the calculated luminosity only indirectly through the value of a parameter which depends on the internal compressibility. We need therefore no equation of transfer of energy for the interior, no assumption about radiative or convective equilibrium in the interior, and no assumption about internal opacity or source-distribution or equation of state.

The resulting formula for the luminosity of a star accounts generally for a number of features associated with the observed masses and luminosities of the stars, like that calculated from the "standard model," of which it may be considered a generalisation and refinement. It gives the observed order of magnitude of the luminosity of a star, but with mean extra-photospheric opacity playing the part taken by mean, or central, or interior opacity in the theory derived from the standard model. If the photospheric opacity and molecular weight are supposed known from spectroscopic studies, the formula gives the numerical value of a parameter which is in some way a measure of the internal compressibility or relative density-distribution. (For the standard model this parameter is unity.) The formula for luminosity is explicitly independent of effective temperature, save to the extent to which this influences the photospheric opacity. For stars of not widely different internal compressibility or relative density-distribution, it gives the proportionality $\bar{\kappa}_1 L \propto M^3$, approximately, which seems to account for the general features of the observed mass-luminosity law.

This substantial dependence of L on M only is not, as it might be supposed, a case of the Russell-Vogt theorem, for the present analysis is independent of the physical nature of the energy-generating process. The Russell-Vogt theorem shows that when the energy-generating process is specified, all variables such as L and r_1 are functions of mass M only. But such functions must involve the parameters occurring in the physical formula for energy-

generation. Our formula, on the other hand, involves no such parameters, explicitly, though such parameters may indirectly influence our parameter characteristic of internal compressibility.

The determination of the radius r_1 is, however, of a different character. On the equilibrium theory here given, the radius is substantially, *i.e.* practically, indeterminate. That is to say, equilibrium conditions exercise very little control on the radius. What equilibrium considerations prescribe is the amount of energy the given mass must generate for a steady state, and the internal temperatures, etc., must adjust themselves until this rate of energy-production is in fact attained. The only way this adjustment can take place is through the radius. The value taken up by the radius must then depend on the parameters in the physical law of generation of energy, and the radius is thus determined in accordance with the Russell-Vogt theorem. It follows, as on the more special theory of the standard model, that whilst the mass-luminosity law is a phenomenon associated with conditions of mechanical equilibrium, the Russell diagram regularities are associated with the laws of energy-generation. A particular inference from this is that since massive main-sequence stars (B- and O-type stars) have masses and luminosities of the same order of magnitude as the giant M-stars, whilst their radii are of different orders of magnitude, the mechanisms of energy-generation must be different in the two classes of stars.* It is suggested that dwarfs and all main-sequence stars possess generally similar modes of energy-generation, whilst giant stars possess some different mode. Nevertheless, since stars of the different types (excluding white dwarfs) obey practically a single mass-luminosity law, the internal relative density-distributions must be much the same in all cases. It will be noticed that there is nothing in our analysis to preclude the possession by all stars of a dense core, which is accordingly here left an open question. Obedience to a single mass-luminosity law does not guarantee that the stars excluding the white dwarfs are composed of perfect gas—we use no equation of state in the interior.

A by-product of our method is a formula for the gas-pressure in a stellar photosphere independent of evaluation of photospheric opacity. The fact that this gives photospheric pressures of the observed order of magnitude is a general check on the correctness of the analysis. The formula shows, moreover, why this photospheric pressure is of much the same order of magnitude in all main-sequence stars. It predicts much smaller photospheric pressures in giant stars of low effective temperature, in accordance with their observed more advanced state of ionization for their photospheric temperatures. The formula can be applied to connect the difference of mass between giant and dwarf stars of the same spectral type with the observed difference of effective temperature (giants are cooler for given type). The formula states that approximately $p_1 \propto T^4/M^2$, where p_1 is the photospheric gas-pressure.

The present paper is self-contained and can be read without previous familiarity with the literature of stellar structure. But it actually arose out of H. N. Russell's recent paper † on "Polytropic Index and Photospheric

* I owe this inference to H. H. Plaskett.

† *M.N.*, 97, 127, 1936.

Conditions." At the end of this paper Russell discussed the meaning of the values of n he had found and concluded that "the computed value of n will afford a valuable parameter for description of certain properties of the photosphere—or rather of the relation of these properties to the star as a whole, but it will be less useful as an index of internal constitution." The present paper, which proceeds without making the polytropic assumption for the interior, substantially supports Russell's conclusion and makes it more precise. The method of the present paper is an extension of that used in a recent series of papers * by the author; the results are akin to ideas first put forward by the author in 1929.†

2. *Equilibrium of the Interior.*—Let ρ be the density, P the total pressure, $M(r)$ the included mass at a point distant r from the centre of an *actual* star. It is particularly to be noted that we are not employing any stellar *model*. From the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho, \quad (1)\dagger$$

we eliminate $M(r)$ by means of

$$M(r) = \int_0^r 4\pi\rho r^2 dr,$$

and obtain in the usual way

$$\frac{1}{r^2} \frac{d}{dr} \left[\frac{r^2}{\rho} \frac{dP}{dr} \right] = -4\pi G\rho. \quad (2)$$

Let ρ_c, P_c denote the central values of ρ and P , for this star, and put

$$\rho = \rho_c y, \quad P = P_c z. \quad (3)$$

Then y, z are dimensionless variables reducing to unity at the centre. Put also

$$r = \left(\frac{P_c}{4\pi G\rho_c^2} \right)^{\frac{1}{2}} x, \quad (4)$$

so that x is dimensionless. Then (2) reduces to

$$\frac{1}{x^2} \frac{d}{dx} \left[\frac{x^2}{y} \frac{dz}{dx} \right] = -y. \quad (5)$$

3. The variables y and z will have definite numerical values at each point of the star under consideration. Let z be supposed plotted against y from centre to boundary. The resulting curve is unknown—its shape depends on the unknown compressibility of the material from point to point. We do know, however, certain things about the form of the curve. At the centre of the star, $y = 1$ and $z = 1$, so the curve will pass through the point (1, 1). At the boundary of the star ($\rho = 0$), P takes a definite non-zero value $\S P_0$ depend-

* *M.N.*, 96, 184, 195, 207, 1936.

† *M.N.*, 90, 17, 1929.

‡ For the justification of this equation at and near the boundary, see § 8.

§ Where $P_0 = \frac{1}{3}aT_0^4 = \frac{1}{3}aT_1^4$ (see § 8).

ing on the effective temperature and P_e is large compared with P_0 ; hence at the boundary $y=0$ and z is a very small number, and so the curve passes *very close to the origin* (0, 0). We do not know the form of the curve from this neighbourhood to (1, 1). If we had adopted a model in the form of a complete polytropic distribution, then we should have had for the shape of our curve generalised parabolas of the form $z=y^{1+1/n}$; such curves pass through the origin, where they are tangent to the y -axis. Our actual curve definitely does not pass through the origin, and may differ widely from a polytropic parabola in the interior, even possibly passing to the right of the ordinate $y=1$ if the star contains negative density gradients near the centre. Generally speaking, however, we may assume z and y to increase monotonically, and we therefore ascribe to our curve some such form as is shown in fig. 1, passing from A_0 ($y=0, z=z_0$) to A_e ($y=1, z=1$).

Now suppose that the outer layers of the star are in radiative equilibrium, with a temperature distribution given by

$$T^4 = \frac{1}{2} T_1^4 \left(1 + \frac{3}{2} \tau\right), \quad (6)$$

where T_1 is as above the effective temperature, T the actual temperature at optical depth τ . We shall call the level $\tau = \frac{2}{3}$ the "photosphere." This is the mean optical depth to which we see. Then at $\tau = \frac{2}{3}$, $T = T_1$. Let y_1 and z_1 denote the values of y and z at this depth. It will be represented by some point A_1 on the (z, y) curve. The values of y_1 and z_1 will be very small, and A_1 will accordingly be very near the origin. It is shown in fig. 2, which represents the portion of the (z, y) curve near the origin on an enlarged scale. Now replace the actual (z, y) curve between $y=0, z=z_0$ and $y=y_1, z=z_1$ by an arc which passes through the origin and is tangent to the actual (z, y) curve at A_1 . We shall see later that we must choose this arc to be tangent to the y -axis at the origin. As we have considerable latitude in the choice of this arc, we can arrange it to have contact of a high order with the actual (z, y) curve at A_1 . This arc, together with the remainder of the original (z, y) curve we shall call the "modified (z, y) curve," and we can represent it to any desired degree of approximation by an analytic function $z=f(y)$, satisfying

$$f(0)=0, \quad f(1)=1.$$

The function $z=f(y)$ represents the actual state of affairs only inside the photosphere, and we shall only so use it. Outside the photosphere we shall switch over to the original curve, using separate analysis.

With $z=f(y)$, relation (5) becomes a differential equation in y ,

$$\frac{1}{x^2} \frac{d}{dx} \left[\frac{f'(y)}{y} x^2 \frac{dy}{dx} \right] = -y. \quad (7)$$

We suppose this equation solved with the initial conditions :

$$\text{at } x=0, \quad y=1, \quad \frac{dy}{dx}=0.$$

Call this solution $y=y(x)$, $y(x)$ depending on f . By analogy with Emden functions for $n < 5$ ((7) being a generalisation of Emden's differential equa-

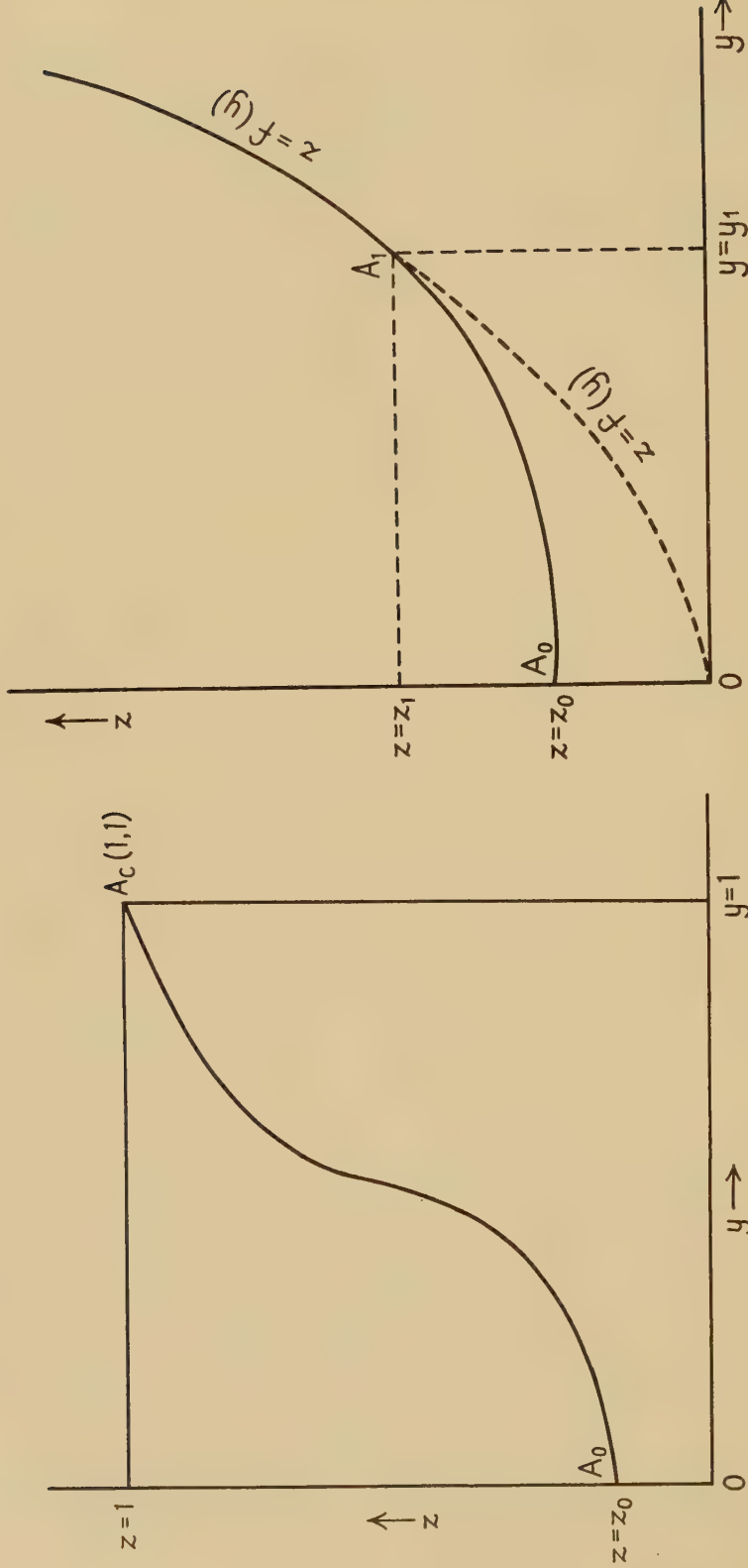


FIG. 1.—Graph of z against y . (N.B. The non-zero boundary value of z , namely z_0 , is much exaggerated for clarity in drawing.)

FIG. 2.—Graphs of z and $f(y)$ near the origin, on an enlarged scale.

tion) we may suppose $y(x)$ to have a zero at some *finite* value x_0 of x . The function $y(x)$ will take the value y_1 at some definite value x_1 of x , and since y_1 is very small, it follows that x_1 will be in the neighbourhood of x_0 . (We shall verify this in detail later (§ 14).) The solution $y = y(x)$ will now reproduce the actual values of y in the range $0 \leq x \leq x_1$; it will not represent them in the range $x_1 < x \leq x_0$. We shall only use it to calculate values of y and z up to $x = x_1$.

4. For later reference we note that when the modified (z, y) curve is that for a complete polytrope, we have $f(y) = y^{1+1/n}$, and if we then put

$$y = \theta^n, \quad z = \theta^{n+1}, \quad x = (n+1)^{\frac{1}{2}} \xi,$$

(7) reduces to Emden's equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left[\xi^2 \frac{d\theta}{d\xi} \right] = -\theta^n. \quad (8)$$

Our transition to a different solution outside $x = x_1$ will represent the departure of the boundary conditions in a real star from those postulated in the usual applications of complete Emden solutions ($T_0 = 0$, $P_0 = 0$).

5. The included mass $M(r)$ is given by

$$M(r) = -\frac{r^2}{G\rho} \frac{dP}{dr} = \frac{P_c^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}} G^{\frac{3}{2}} \rho_c^2} \left[-\frac{x^2}{y} \frac{dz}{dx} \right], \quad (9)$$

where dz/dx denotes $f'(y)dy/dx$. The mean density $\bar{\rho}(r)$ inside r is given by

$$\frac{\bar{\rho}(r)}{\rho_c} = \frac{M(r)}{\frac{4}{3}\pi r^3 \rho_c} = -\frac{3}{xy} \frac{dz}{dx}. \quad (10)$$

6. We are interested in the photospheric values of the physical variables. Let r_1 , ρ_1 , P_1 be the radius, density and total pressure at the photosphere, M the included mass; M will be indistinguishable from the mass of the star. Then

$$\rho_1 = \rho_c y_1, \quad P_1 = P_c z_1. \quad (11)$$

Using these to eliminate the unknown ρ_c and P_c from (4) and (9), and then applying (4) and (9) to the photospheric level $x = x_1$, we have

$$r_1 = \left(\frac{P_1}{4\pi G \rho_1^2} \right)^{\frac{1}{2}} \frac{x_1 y_1}{z_1^{\frac{1}{2}}}, \quad (12)$$

$$M = \frac{P_1^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}} G^{\frac{3}{2}} \rho_1^2} \frac{y_1}{z_1^{\frac{3}{2}}} \left[-x^2 \frac{dz}{dx} \right]_{x=x_1}. \quad (13)$$

7. Now use the fact that at the photosphere the material is gaseous. Let p_1 be the gas-pressure, μ_1 the mean molecular weight, and let β_1 be the ratio of gas-pressure to total pressure, all at the photosphere. Then

$$p_1 = (R/\mu_1) \rho_1 T_1, \quad (14)$$

$$\beta_1 = p_1/P_1. \quad (15)$$

Eliminating P_1 and ρ_1 from (12) and (13) by means of (14) and (15), we have

$$r_1 = \frac{(R/\mu_1)T_1}{(4\pi G\beta_1 p_1)^{\frac{1}{2}}} \frac{x_1 y_1}{z_1^{\frac{3}{2}}} \quad (16)$$

$$M = \frac{(R/\mu_1)^2 T_1^2}{(4\pi)^{\frac{1}{2}} G^{\frac{3}{2}} \beta_1^{\frac{3}{2}} p_1^{\frac{1}{2}}} \frac{y_1}{z_1^{\frac{3}{2}}} \left[-x_1^2 \frac{dz}{dx} \right]_1. \quad (17)$$

The last equation determines p_1 in the form

$$p_1 = \frac{(R/\mu_1)^4 T_1^4}{4\pi G^3 M^2 \beta_1^3} \frac{y_1^2}{z_1^3} \left[-x_1^2 \left(\frac{dz}{dx} \right)_1 \right]^2 \quad (18)$$

and division of (16) by (17) determines r_1 in the form

$$r_1 = \frac{GM\beta_1}{(R/\mu_1)T_1} \left[-\frac{z_1}{x_1 (dz/dx)_1} \right]. \quad (19)$$

The occurrence of the power T_1^4 in (18) has nothing to do with radiation-pressure.

The value of x_1 is unknown. If it is possible to eliminate it between (18) and (19), we obtain a relation between p , M and r_1 . Before considering the details of this elimination we shall determine another expression for p_1 by working inwards instead of outwards.

8. *Equilibrium of the Extra-photospheric Layers.*—We shall ignore the complications caused by the reversing layer and chromosphere, and consider simply the equilibrium of the layers above $\tau = \frac{2}{3}$ as affected by gravity and non-selective radiation-pressure.

We must be careful to ensure that the approximation by which the transfer of momentum due to radiation is replaced by a hydrostatic pressure remains valid near the boundary, and that it is consistent with the usual approximation to the temperature-distribution for radiative equilibrium. If L is the luminosity, the net flux is $L/4\pi r_1^2$, and the force on the element ρdr due to absorption of radiation is accurately

$$\frac{L}{4\pi r_1^2} \frac{\kappa \rho dr}{c},$$

κ being the mass-absorption coefficient. The re-emission of this radiation makes no contribution. The equation of mechanical equilibrium can accordingly be written

$$\frac{dp}{dr} - \frac{\kappa \rho}{c} \frac{L}{4\pi r_1^2} = -g\rho. \quad (20)$$

Further,

$$\frac{d\tau}{dr} = -\kappa\rho. \quad (21)$$

Hence, to the extent to which (6) represents the temperature-distribution, we have on differentiation,

$$\frac{dT^4}{dr} = -\frac{3}{4}T_1^4\kappa\rho.$$

But

$$L = \pi a c r_1^2 T_1^4. \quad (22)$$

Hence,

$$-\frac{\kappa \rho}{c} \frac{L}{4\pi r_1^2} = \frac{d(\frac{1}{3}aT^4)}{dr}.$$

Accordingly, even near the boundary, we may write (20) in the form (1) with

$$P = p + \frac{1}{3}aT^4.$$

Now let T_0 be the boundary temperature, $P_0 = \frac{1}{3}aT_0^4 = \frac{1}{6}aT_1^4$ the boundary value of P . From (1) and (21),

$$\frac{d\tau}{dP} = \frac{\kappa}{g},$$

whence

$$\frac{2}{3} = \tau_1 = \int_0^{\tau_1} d\tau = \int_{P_0}^{P_1} \frac{\kappa}{g} dP = \frac{\bar{\kappa}_1}{g} (P_1 - P_0), \quad (23)$$

where the mean extra-photospheric opacity $\bar{\kappa}_1$ is defined by

$$\bar{\kappa}_1 = \frac{1}{P_1 - P_0} \int_{P_0}^{P_1} \kappa dP = \frac{\int_{r_1}^{\tau_0} \kappa \rho dr}{\int_{r_1}^{\tau_0} \rho dr}.$$

But

$$P_1 = p_1 + \frac{1}{3}aT_1^4,$$

so that

$$P_1 - P_0 = p_1 + \frac{1}{6}aT_1^4.$$

Hence, by (23),

$$p_1 = \frac{2}{3} \frac{g}{\bar{\kappa}_1} - \frac{1}{6}aT_1^4.$$

But

$$g = \frac{GM}{r_1^2} = \frac{GM\pi acT_1^4}{L}.$$

Hence,

$$p_1 = \frac{1}{6}aT_1^4 \left[\frac{4\pi cGM}{\bar{\kappa}_1 L} - 1 \right]. \quad (24)$$

This is the value of the photospheric pressure obtained by working inwards. Incidentally we have now determined β_1 , for

$$\beta_1 = \frac{p_1}{p_1 + \frac{1}{3}aT_1^4} = \frac{1 - \frac{\bar{\kappa}_1 L}{4\pi cGM}}{1 + \frac{\bar{\kappa}_1 L}{4\pi cGM}}. \quad (25)$$

This may also be written

$$L = \frac{4\pi cGM}{\bar{\kappa}_1} \frac{1 - \beta_1}{1 + \beta_1}, \quad (26)$$

a formula for luminosity which fairly accurately takes into account the non-zero boundary value of T .

9. *Value of the Luminosity.*—The pressure must be continuous at the photosphere; hence the two expressions (18) and (24) for p_1 must be equal. Equating them, we see that T_1^4 divides out and we are left with

$$\frac{1}{6}a \left[\frac{4\pi cGM}{\bar{\kappa}_1 L} - 1 \right] = \frac{(R/\mu_1)^4}{4\pi G^3 M^2} \frac{\left(1 + \frac{\bar{\kappa}_1 L}{4\pi cGM} \right)^3}{\left(1 - \frac{\bar{\kappa}_1 L}{4\pi cGM} \right)^3} \frac{y_1^2}{z_1^3} \left[-x_1^2 \left(\frac{dz}{dx} \right)_1 \right]^2. \quad (27)$$

This can be rewritten as

$$\bar{\kappa}_1 L = \frac{8\pi^2 c G^4 (\frac{1}{3}a) M^3}{(R/\mu_1)^4} \frac{\left(1 - \frac{\bar{\kappa}_1 L}{4\pi cGM} \right)^4}{\left(1 + \frac{\bar{\kappa}_1 L}{4\pi cGM} \right)^3} \frac{z_1^3}{y_1^2} \left[-x_1^2 \left(\frac{dz}{dx} \right)_1 \right]^{-2}. \quad (28)$$

Again, (19) may be written

$$\left(\frac{L}{\pi a c T_1^4} \right)^{\frac{1}{2}} = r_1 = \frac{GM}{(R/\mu_1) T_1} \cdot \frac{1 - \frac{\bar{\kappa}_1 L}{4\pi cGM}}{1 + \frac{\bar{\kappa}_1 L}{4\pi cGM}} \cdot \left[-\frac{z_1}{x_1 (dz/dx)_1} \right], \quad (29)$$

from which pair of equations we may solve for r_1 and T_1 , obtaining

$$r_1 = \left(\frac{\pi a c}{L} \right)^{\frac{1}{2}} \frac{G^2 M^2}{(R/\mu_1)^2} \frac{\left(1 - \frac{\kappa_1 L}{4\pi cGM} \right)^2}{\left(1 + \frac{\bar{\kappa}_1 L}{4\pi cGM} \right)^2} \frac{z_1^2}{x_1^2 (dz/dx)_1^2}, \quad (30)$$

$$T_1 = \left(\frac{L}{\pi a c} \right)^{\frac{1}{2}} \frac{R/\mu_1}{GM} \frac{1 + \frac{\bar{\kappa}_1 L}{4\pi cGM}}{1 - \frac{\bar{\kappa}_1 L}{4\pi cGM}} \left[-\frac{x_1 (dz/dx)_1}{z_1} \right]. \quad (31)$$

The relation (27) can also be written in the form *

$$4\pi (\frac{1}{3}a) G^3 M^2 (\mu_1/R)^4 \frac{\beta_1^4}{1 - \beta_1} = \frac{y_1^2}{z_1^3} \left[-x_1^2 \left(\frac{dz}{dx} \right)_1 \right]^2 \quad (32)$$

which is a quartic equation for β_1 , whose solution yields L on use of (26).

All these formulæ contain the very small numbers y_1 and z_1 , functions of the unknown x_1 . Our object is now to put the coefficients involving x_1 in finite form and to eliminate x_1 .

10. Since y and z possess zeros at $x = x_0$ as calculated from (7), they will

* This formula is essentially due to H. Vogt, *Veröff. Jena*, Nr. 3, 1930. See also his more recent paper, *Veröff. Heidelberg*, Nr. 5, 1936.

be representable near $x = x_0$ by series of ascending, not necessarily integral powers of $(x_0 - x)$, say

$$y = A(x_0 - x)^s + \dots, \quad (s, s' > 0). \quad (33)$$

$$z = B(x_0 - x)^{s'} + \dots \quad (34)$$

Inserting these expansions in (5), we find that the lowest power of $(x_0 - x)$ on the left-hand side is

$$\frac{B}{A} s'(s' - s - 1)(x_0 - x)^{s' - s - 2},$$

provided $s'(s' - s - 1)$ is not zero. That on the right-hand side is

$$-A(x_0 - x)^s.$$

Hence if $s'(s' - s - 1)$ is not zero, we must have

$$s' - s - 2 = s,$$

and

$$\frac{B}{A} s'(s' - s - 1) = -A.$$

These give $s' = 2(s + 1)$, and then

$$B = -\frac{\frac{1}{2}A^2}{(s + 1)^2}.$$

But B must be positive, since P and z are positive. It follows that $s'(s' - s - 1)$ must be zero. But s' cannot be zero, since by construction of the function $z = f(y)$, z vanishes at $x = x_0$. Hence

$$s' = s + 1. \quad (35)$$

A check on this result is afforded by (10). For $\bar{\rho}/\rho_c$ must be a finite number not vanishing with $x_0 - x$, and with expansions (33) and (34) with $s' = s + 1$ we find in fact near x_0 ,

$$\frac{\bar{\rho}(r)}{\rho_c} \sim \frac{3B(s + 1)}{Ax_0},$$

a finite number.

The value of x_0 , and the values of A and B , depend on the complete march of the function $f(y)$ from $y = 1$ to $y = 0$, *i.e.* on the complete march of compressibility from centre to boundary.

11. We now notice that this exponent s is just the polytropic index for the photospheric layers. For by (33) and (34) we must have just inside the photosphere

$$z \sim B \left(\frac{y}{A} \right)^{(s+1)/s}$$

or

$$\frac{P}{P_c} \sim B \left(\frac{\rho}{A\rho_c} \right)^{1+1/s}.$$

12. The coefficients in (28), (30) and (31) can now be evaluated. Applying (33) and (34) to the argument $x = x_1$, we find for the dominant terms

$$\frac{y_1^2}{z_1^3} \left[-x_1^2 \left(\frac{dz}{dx} \right)_1 \right]^2 \sim (s+1)^2 \frac{A^2}{B} x_0^4 (x_0 - x_1)^{s-3}, \quad (36)$$

and

$$-\frac{x_1}{z_1} \left(\frac{dz}{dx} \right)_1 \sim \frac{(s+1)x_0}{x_0 - x_1}. \quad (37)$$

Hence the approximate forms of (28), (30) and (31) are

$$\bar{\kappa}_1 L \sim \frac{8\pi^2 c G^4 (\frac{1}{3}a) M^3}{(R/\mu_1)^4} \frac{\left(1 - \frac{\bar{\kappa}_1 L}{4\pi c G M}\right)^4}{\left(1 + \frac{\bar{\kappa}_1 L}{4\pi c G M}\right)^3} \frac{B}{(s+1)^2 A^2 x_0^4} (x_0 - x_1)^{3-s}, \quad (28')$$

$$r_1 \sim \left(\frac{\pi a c}{L}\right)^{\frac{1}{2}} \frac{G^2 M^2}{(R/\mu_1)^2} \frac{\left(1 - \frac{\bar{\kappa}_1 L}{4\pi c G M}\right)^2}{\left(1 + \frac{\bar{\kappa}_1 L}{4\pi c G M}\right)^2} \frac{(x_0 - x_1)^2}{(s+1)^2 x_0^2}, \quad (30')$$

$$T_1 \sim \left(\frac{L}{\pi a c}\right)^{\frac{1}{2}} \frac{R/\mu_1}{G M} \frac{1 + \frac{\bar{\kappa}_1 L}{4\pi c G M}}{1 - \frac{\bar{\kappa}_1 L}{4\pi c G M}} \frac{(s+1)x_0}{x_0 - x_1}. \quad (31')$$

13. If $s \neq 3$, $\bar{\kappa}_1 L$ depends on $(x_0 - x_1)$. We may then eliminate $(x_0 - x_1)$ between the approximations (28') and (30'), obtaining a relation between L , M and r_1 involving $\bar{\kappa}_1$, μ_1 and that aspect of the internal density-distribution which is summed up under the factor B/A^2 . But if $s=3$, relation (28') is approximately independent of $(x_0 - x_1)$, and yields as it stands a relation between $\bar{\kappa}_1 L$, M and μ_1 independent of r_1 . When $s \neq 3$, we can calculate r_1 for arbitrary L and M ; but this is not true in the exceptional case $s=3$.

14. Now it is known from the researches of Woltjer,* Jeans and others that just below the photosphere the ratio of radiation-pressure to gas-pressure soon settles down to a very slowly changing function. As a first approximation suitable as a basis of discussion, we may take the ratio to be approximately constant with depth. That is, in these layers we may take $p \propto \rho T \propto T^4$, which gives $P \propto \rho^{\frac{4}{3}}$. Thus to a first approximation the polytropic index s for the photospheric layers is about 3. The exceptional case $s=3$ is in fact precisely the one that presents itself in practice under the actual conditions of a stellar atmosphere. We can therefore write (28') for an actual star in the form

$$\bar{\kappa}_1 L \sim \frac{8\pi^2 c G^4 (\frac{1}{3}a) M^3}{(R/\mu_1)^4} \frac{\left(1 - \frac{\bar{\kappa}_1 L}{4\pi c G M}\right)^4}{\left(1 + \frac{\bar{\kappa}_1 L}{4\pi c G M}\right)^3} \frac{B}{(3+1)^2 A^2 x_0^4}; \quad (38)$$

* Woltjer, *B.A.N.*, **2**, 171, 1924; Jeans, *M.N.*, **85**, 199, 1925. Their analysis has been repeated in slightly different forms by the author in *M.N.*, **85**, 784, 1925, and *Phil. Trans.*, **228**, A, 456, 1929; and by Chandrasekhar, *M.N.*, **92**, 188, 1932.

whilst (32) takes the form

$$4\pi\left(\frac{1}{3}a\right)G^3M^2(\mu_1/R)^4\frac{\beta_1^4}{1-\beta_1}=\frac{(3+1)^2A^2x_0^4}{B}. \quad (39)$$

15. *Orders of Magnitude.*—To see the order of magnitude of the factor involving B/A^2 we proceed as follows. If the internal compressibility happens to be polytropic, we have

$$\frac{y_1^2}{z_1^3}\left[-x_1^2\left(\frac{dz}{dx}\right)_1\right]^2=(n+1)^3\theta_1^{n-3}\xi_1^4\left(\frac{d\theta}{d\xi}\right)_1^2.$$

Hence, if $n=3$ as well as $s=3$, we have

$$(3+1)^2\frac{A^2}{B}x_0^4=(3+1)^3\left[\xi_0^4\left(\frac{d\theta}{d\xi}\right)_0\right]_{n=3}=64\times(2.018\dots)^2.$$

If therefore we write

$$\alpha=\frac{64\times(2.018\dots)^2}{(3+1)^2x_0^4(A^2/B)}, \quad (40)$$

we may assume that in a range of cases α is of the order of magnitude of unity, though it will only be exactly unity when the material inside the photosphere constitutes a truncated polytrope of index $n=3$.

Inserting now the numerical values

$$c=3\times 10^{10}, \quad a=7.64\times 10^{-15}, \quad G=6.66\times 10^{-8}, \quad R/\mu_H=8.26\times 10^7, \\ M_\odot=1.985\times 10^{33}, \quad L_\odot=3.78\times 10^{33},$$

we see that (38) gives

$$\frac{L}{L_\odot}=\left(\frac{M}{M_\odot}\right)^3\left(\frac{\mu_1}{\mu_H}\right)^4\left(\frac{100}{\bar{\kappa}_1}\right)\left(\frac{1-\frac{\bar{\kappa}_1L}{4\pi cGM}}{1+\frac{\bar{\kappa}_1L}{4\pi cGM}}\right)^4\times 0.20\alpha, \quad (41)$$

(39) gives

$$1-\beta_1=\left(\frac{M}{M_\odot}\right)^2\left(\frac{\mu_1}{\mu_H}\right)^4\beta_1^4\times 0.00305\alpha, \quad (42)$$

and (31') gives

$$\frac{x_0-x_1}{x_0}=\left(\frac{L}{L_\odot}\right)^{\frac{1}{2}}\frac{M_\odot}{M}\frac{\mu_H}{\mu_1}\frac{1+\frac{\bar{\kappa}_1L}{4\pi cGM}}{1-\frac{\bar{\kappa}_1L}{4\pi cGM}}\frac{5.72}{T_1}. \quad (43)$$

Again (18) gives with the same data

$$p_1=\left(\frac{T_1}{1000}\right)^4\left(\frac{M_\odot}{M}\right)^2\left(\frac{\mu_H}{\mu_1}\right)^4\left(\frac{1+\frac{\bar{\kappa}_1L}{4\pi cGM}}{1-\frac{\bar{\kappa}_1L}{4\pi cGM}}\right)^3\times\frac{0.836}{\alpha}\text{ dynes, cm.}^{-2}. \quad (44)$$

These numerical formulæ show at once that we have successfully hit off the orders of magnitude of the various quantities. Formula (43) shows that

since T_1 is of the order of several thousand, $(x_0 - x_1)/x_0$ is a very small number, "usually less than one part in 1000," as Russell remarked. This is a measure of the thickness, compared with the stellar radius, of the fictitious extra-photospheric layer which would be computed by taking $T_0 = 0$. The actual thickness outside the photosphere must exceed this.

Formula (44) gives the right order of magnitude for the photospheric gas-pressure. For the Sun, for example, for which $T_1 = 5740^\circ$, taking $\mu_1 = \mu_H$, it gives $p_1 = (900/\alpha)$ dynes cm.⁻², or for $\alpha \sim 1$ say $p_1 \sim 10^{-3}$ atmos. It should be remembered that p_1 is the total gas-pressure, not the partial electron-pressure of ionization theory. A physical determination of p_1 clearly affords a method for evaluating α and so finding something about the internal compressibility.

Lastly, since $\bar{\kappa}_1$ is of the order of magnitude of 100 for the Sun,* assuming $\alpha \sim 1$, formula (41) gives the right order of magnitude for the Sun's luminosity. More accurately, we may say that (41) indicates that α is of the order of magnitude of unity for the Sun, though it suggests that α may be as much as 5. The difference would be due to departure of the Sun's interior from the density-distribution characteristic of a polytrope $n = 3$. It is clear that (41) gives generally the right orders of magnitude for the luminosities of all stars, owing to the dominating character of the factor M^3 . We may notice that we have here further evidence that the value of s for actual photospheres must be close to $s = 3$. For by (28'), if s were appreciably greater than 3, the calculated value of L would come out very large; if appreciably smaller than 3, very small.

Applications

16. *Photospheric Pressure.*—Several inferences of general interest may be made from the above formulæ. In the first place, (44) or (18) gives approximately, for stars of similar internal compressibility and atmospheric composition,

$$p_1 \propto T_1^4/M^2. \quad (45)$$

This accounts for the general similarity of order of magnitude of the photospheric pressure in the stars in general. Considering, for example, main sequence stars only, an increase of T_1 from 3000° to $18,000^\circ$ is accompanied by an increase of mass from say $0.5\odot$ to $10\odot$. In this range the factor T_1^4/M^2 changes only in the ratio $6^4/20^2 = 3.24$. Again, (45) gives much smaller pressures in the photospheres of giant stars of large mass as compared with main sequence (dwarf) stars of the same effective temperature. It can readily be applied to determine the mass-ratio of a giant and dwarf of the same spectral type, by assuming that p_1 , the photospheric gas-pressure, is proportional to $T_1^{\frac{5}{2}}e^{-\chi/kT_1}$, where χ is the mean level of ionization potential. We then have

$$\left[\frac{T_1^{\frac{5}{2}}e^{-\chi/kT_1}}{T_1^4/M^2} \right]_{\text{giant}} = \left[\frac{T_1^{\frac{5}{2}}e^{-\chi/kT_1}}{T_1^4/M^2} \right]_{\text{dwarf}}.$$

* Chandrasekhar, *loc. cit.*, Table II.

A knowledge of the temperature-difference between giants and dwarfs of the same spectral type then gives the mass-ratio. A preliminary calculation of this kind, taking $\chi = 6$ electron-volts, $(T_1)_g = 3000^\circ$, $(T_1)_d = 4000^\circ$ gives $M_g/M_d = 42$; if M_d is taken as $0.5\odot$, this gives $M_g = 21\odot$, which seems about right for the assumed temperature-difference. The main point is that in principle the proportionality (45) affords a method of calculating from spectroscopic evidence the mass of a single giant star by comparing it with a dwarf of known mass of the same type, the importance of which in connection with spectroscopic parallaxes was first commented on by Pannekoek.*

17. *The Mass-luminosity Law.*—Again, for stars of approximately constant photospheric opacity $\bar{\kappa}_1$ and approximately similar internal density-distribution, (41) gives $L \propto M^3$. Variations in $\bar{\kappa}_1$ may be expected to be important, and will effectively modify the exponent of M , but the approximate observed proportionality of the luminosities of the stars to the cubes of their masses is well known. It was remarked upon by Jeans in 1928.† Formula (41) appears to give the leading term in the observed behaviour.

18. *Determination of Internal Density-distribution.*—In formula (41), L , M , μ_1 and $\bar{\kappa}_1$ are all in principle capable of direct observation. The molecular weight μ_1 can be determined from the spectroscopic analysis of stellar atmospheres, following Russell, Unsöld and others, and then, the composition being known, $\bar{\kappa}_1$ could be calculated from Pannekoek's tables,‡ which give essentially physical determinations of κ under known conditions. Accordingly, a determined set of L , M , μ_1 , $\bar{\kappa}_1$ for a particular star yield the value of α for this star, *i.e.* the value of $A^2x_0^4/B$. A knowledge of $A^2x_0^4/B$ imposes a restriction on the function $z=f(y)$, from whose behaviour A and B are determined, and so tells us something of the compressibility of the interior, *i.e.* something of the internal density-distribution. It is not difficult to express this restriction in explicit algebraic form, using the properties of (7), but the account of this is deferred to a future occasion. The point of principle is that a value for α appears to be the utmost that can be attained by equilibrium analysis alone, without using some particular stellar model and without assuming a law of energy-generation. Of course if we adopt a particular model, as, for example, in Eddington's theory, we can express α in terms of the parameters defining the model, and then, conversely, evaluate one of these parameters from a knowledge of α . This, however, tells us no new fact about the stellar interior unless the model can be justified on other grounds; we cannot, for example, *infer* an opacity or molecular weight for the interior which is not simply an extrapolation from the photosphere. In Eddington's theory α is taken to be unity. But in his theory the precise significance of the various relations is obscured by the fact that both $n=3$ and $s=3$; consequently the extra-photospheric opacity and photospheric molecular weight, which are actually involved, have been mistakenly confused with the interior opacity and molecular weight; the only part these play is in determining the interior compressibility, but whatever their values, if we adopt $n=3$ the

* *Observatory*, 46, 304, 1923.

† *Astronomy and Cosmogony*, p. 126.

‡ *Pub. Astron. Inst. Amsterdam*, No. 4, 1935.

value of α is unity. Thus we have at present no knowledge of the chemical composition of stellar *interiors*. Still less can we assert that they are in a state of perfect gas throughout.

19. *Fixation of Radius by the Energy-generation Law.*—When $s=3$, the radius r_1 appears incapable of determination by the formulæ we have given, given only L , M and $f(y)$; for x_1 is unknown. It is true, of course, that even when $s=3$, closer approximations to the dimensionless factor in (28) will involve $x_0 - x_1$, so that the elimination of $x_0 - x_1$ between (30) and (28) can still in principle be performed, yielding a relation between L , M and r_1 . But when $s=3$, the control on r_1 exerted by this relation will be very slight; r_1 will be *practically* indeterminate from equilibrium considerations alone. What then determines r_1 for an actual star when $s=3$? It is clear that the control must be exerted by the energy-generating process. When $s=3$, (28) effectively determines a value of L (given $\bar{\kappa}_1$, μ_1 and M), which somehow or other the star has got to generate. It must therefore assume such an internal state that, compatibly with the law of generation under these physical conditions, this amount L is in fact produced. The only adjustment the star has at its disposal* is its radius, and with it the internal temperatures and the absolute values of the internal densities. This is still true if there is no one-way energy-generating mechanism, but only an excess of reversible processes in a certain direction controlled by the surface losses.

The luminosity formulæ here derived from equilibrium considerations are essentially formulæ determining the surface losses of energy by a given mass with a given *relative* internal density-distribution. This is the only sense in which we can speak of a luminosity formula derived from equilibrium considerations alone, in the absence of the adoption of a specific mode of energy-generation. Thus though our formulæ express L as a function of M independent of r_1 (when $s=3$), they are not a case of the Russell-Vogt theorem. The latter theorem shows that for a given law of generation of energy, L is a function of M . But this function will involve the parameters occurring in the physical energy-generation law. Our luminosity formula, however, is explicitly independent of the mode of generation of energy, though it may depend implicitly on the latter through α . This shows that the ground we have covered is different from that covered by the Russell-Vogt theorem.

I am indebted to Professor Russell for kindly sending me a copy of his paper in advance of publication. Also I owe to Professor H. H. Plaskett several suggestions as to the form of presentation.

20. *Summary.*—For a star of any structure, the pressure at the photosphere is calculated in terms of mass, photospheric molecular weight, effective temperature and a coefficient depending on the form of the internal relative density-distribution. The photospheric pressure is also calculated from the equilibrium of the extra-photospheric layers in terms of mass, luminosity, effective temperature and photospheric opacity. The equality of the two values of the pressure yields a formula independent of effective temperature connecting luminosity with mass. The coefficient in this formula is analysed

* For a given α , that is. Processes which affect r_1 may affect α also, but changes in α may be expected to be of a secondary nature. $\bar{\kappa}_1$ may also be affected.

into factors depending respectively on the photospheric compressibility and the internal compressibility. The photospheric compressibility can be approximately represented by a polytropic index $s=3$. The resulting luminosity formula appears to represent the main features of the observed mass-luminosity correlation and gives the right order of magnitude for the luminosity. It is shown what inferences are possible about a stellar interior from observable data. The pressure in a stellar photosphere, for stars of similar internal structure, should vary as the fourth power of the effective temperature and inversely as the square of the mass, independent of photospheric opacity. Sundry applications of this result are made.

University Observatory, Oxford :

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VISUAL OBSERVATIONS OF INTENSITY OF DARK MARKINGS ON THE SUN, AND OF PROMINENCES.

R. v. d. R. Woolley and H. W. Newton.

(Communicated by the Astronomer Royal)

1. In a previous communication to the Society* we have described visual observations of the contour of the line $H\alpha$ with a spectrohelioscope, using a special auxiliary photometer. While we recognise that the spectrohelioscope is not an ideal instrument for the determination of line contours, it is obvious that it offers special advantages for the determination of the spectral contours of transient phenomena, namely, prominences on the limb and bright and dark markings on the disk, and we have felt justified in continuing measurements of this kind with the spectrohelioscope.

A line contour determined with any spectrograph requires correction (*a*) for the finite resolving power of the instrument and (*b*) for what may be called heterochromatic light, unless a monochromator has been used. The presence of light whose wave-length differs from that of the line by an amount large compared with the resolution of the instrument is a well-known phenomenon. (A discussion of the subject will be found in *The Observatory*, 58, 5, 1935.) The effects of finite resolving power may be computed if the effective resolving power of the spectrograph is known. To eliminate heterochromatic light it is desirable to use a monochromator. Failing this, it is desirable to determine the amount of heterochromatic light experimentally. In our case it has not been possible to erect a monochromator or to make a direct determination of the heterochromatic light, and we have instead determined this amount by comparison of our contour for $H\alpha$ (corrected for finite resolving power) with Thackeray's contour† which was determined with a higher resolution than ours and with a monochromator.

2. *The Correction for Finite Resolving Power.*—Consider first an imaginary spectrograph of infinite resolving power. All the light of wave-length λ

* *M.N.*, 96, 5, 1935.

† *M.N.*, 95, 293, 1935.

falls on a line in the focal plane which has some co-ordinate x , so that the intensity distribution $I(x)$ is an accurate representation of the intensity distribution $I(\lambda)$ in the spectrum under measurement. In a real instrument of finite resolving power, the line x receives a certain amount of light from wave-lengths contiguous to the wave-length which properly corresponds to x (and loses some of its own light), so that the measured intensity $I_1(x)$ will be given by

$$I_1(x) = \int I(x) f(\xi - x) d\xi, \quad (1)$$

where the function $f(\xi - x)$ depends on the resolving performance of the spectrograph. Its general form is that it has a maximum at $\xi - x = 0$ and that it vanishes when $|\xi - x|$ attains values larger than the practical resolution of the instrument in the x co-ordinate.

In the special case where the lack of resolution is *entirely* due to the use of a wide slit (of width $2h$) we have

$$\begin{aligned} f(\xi - x) &= 1 & |\xi - x| < h \\ f(\xi - x) &= 0 & |\xi - x| \geq h, \end{aligned}$$

and the intensity received by the point x takes the simple form

$$I_1(x) = \frac{1}{2h} \int_{x-h}^{x+h} I(x) dx. \quad (2)$$

Since the slits used in the spectrohelioscope were about 0.1 mm. wide, corresponding to 0.5 Å., while the resolving power of the grating is 43,000, the lack of resolution may in this case be considered as being due to the wide slits alone. We proceed on that assumption, using equation (2).

In the spectrohelioscope the spectrum is viewed through a second slit. The effect of this second slit as regards resolution is similar to the first, and the intensity $I_2(x)$ which is transmitted by the second slit is given by

$$I_2(x) = \frac{1}{2h} \int_{x-h}^{x+h} I_1(x) dx. \quad (3)$$

Now let us consider equation (2). If $I(x)$ is a constant or is a linear function of x , we shall have $I_1(x) = I(x)$; it is only when the second differences of $I(x)$ do not vanish that $I_1(x) \neq I(x)$. The wide slit is sampling a set of values of a thing (the intensity of the spectrum), and if the thing is linear the mean value is the value in the middle of the range sampled. Only second (and fourth and sixth, etc.) differences upset this. It is only a matter of algebra to develop expressions for $I_1(x)$ and $I_2(x)$ when $I(x)$ has any finite set of differences, but we have found in practice that the fourth differences have no practical significance in our contours (or Thackeray's), and we have accordingly only worked with the second difference. This gives an extremely simple arithmetical procedure for correcting a contour for finite resolving power, as will appear.

If the fourth and higher differences of $I(x)$ may be neglected, then

$$I_1(x) = I(x) + \frac{1}{3!} h^2 \cdot I''(x) \quad I_2(x) = I(x) + \frac{2}{3!} h^2 I''(x).$$

Conversely, neglecting the fourth and higher differences of $I_2(x)$,

$$I(x) = I_2(x) - \frac{2}{3!} h^2 I_2''(x). \tag{4}$$

It will be noticed that the correction for finite resolving power does not alter the equivalent breadth of the line, since

$$\int_{-\infty}^{\infty} I(x) dx = \int_{-\infty}^{\infty} I_2(x) dx.$$

Equation (4) gives a very convenient way of constructing $I(x)$, the true contour, from $I_2(x)$, the observed contour.

Plot the observed contour $I_2(x)$ and read off a number of values of the intensity for equally spaced values of the abscissa x , choosing this spacing to be a fraction such as one-fifth or one-sixth of the slit width $2h$. Then the second differences of the ordinates read from the graph will give approximate values for $I_2''(x)$. The contour $I_2(x)$ can be corrected to the true contour $I(x)$ by using values of $I_2''(x)$ obtained in this way (and smoothed where necessary) the units in which h is expressed in equation (4) being the unit of spacing of the abscissa. Table I shows an example: the central portion of $H\alpha$ as observed by Newton and Barton. Here $h = 2.5$.

TABLE I
Correction of Contour of Normal $H\alpha$ for Finite Resolving Power

Distance in Angstroms from Centre	$I_2(x)$	1st Difference	2nd Difference	2nd Difference Smoothed	$I(x)$
+ 1.0	67½				
		3			
0.9	64½		0	- 0.5	65½
		3			
0.8	61½		- 1	- 0.5	62½
		4			
0.7	57½		- ½	- 0.9	59½
		4½			
0.6	53		- 1	- 0.9	55
		5½			
0.5	47½		- 2	- 0.5	48
		7½			
0.4	40		0	0.0	40
		7½			
0.3	32½		+ 2½	+ 0.7	31
		5			
0.2	27½		+ 2½	+ 1.5	24½
		2½			
0.1	25		+ ½	+ 1.8	21
		2			
0.0	23		+ 2½	+ 1.7	19
		- ½			
- 0.1	23½		+ 1½	+ 1.6	20
		- 2			
- 0.2	25½				

The unit of $I(x)$ is such that the continuous background outside the line = 100.

The smoothed values are the means of consecutive fives. Half-integers are shown in order to illustrate the correction carefully, but no confidence is to be placed in them from the point of view of probable error.

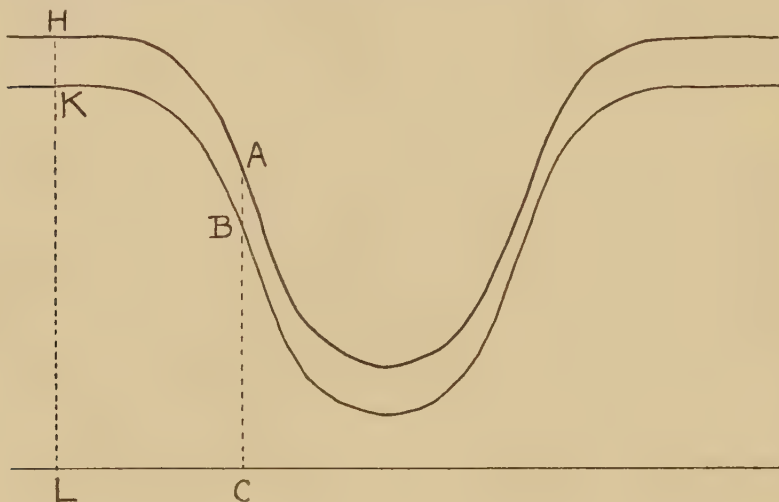


FIG. 1.

3. *Heterochromatic Light*.—Suppose that at any point which is clear of absorption lines the spectrograph overlays the spectrum with the fraction s of heterochromatic light. In fig. 1 the upper curve represents the actual spectrum given by the spectrograph, and the lower curve represents the spectrum which would be given if a monochromator was placed in front of the spectrograph. Then $HK = s \cdot KL$. Consider a point inside an absorption line. The intensity given by the actual spectrograph is $r' = AC/HL$, and the true intensity r is BC/KL . Then

$$r = r' - s(1 - r').$$

As will be seen from Table I, the value of r' at the centre of the line is 0.19. If we assume that the true central intensity of $H\alpha$ on the undisturbed disk is 0.15 (following Thackeray), then $s = 0.05$. This is a very reasonable result, as another grating examined at Greenwich by one of us and Mr. Davidson gave precisely this value of s . In neither case is the result supposed to be exact—the 5 per cent. might very well be 6 or 4, and the two gratings need not give the same result.

4. Measurements of the contour of $H\alpha$ on the undisturbed disk have been described in our previous paper. The contour of $H\alpha$ in dark markings has now been determined in the following manner. A calibrated wedge, incorporating a marginal strip of clear glass of the same thickness as that of the wedge itself, is inserted in the field of view between the eye-piece and the rotating prisms of the spectrohelioscope. The dark $H\alpha$ marking is seen through the clear glass and the neighbouring background, in the same wave-length (in the middle of $H\alpha$ if the marking has no radial velocity) is seen through the graduated wedge. The wedge is then adjusted until

a match in intensity is made at the fiducial point in the field between the marking and the background, and a scale reading is taken. The line-shifter is then turned so that the wave-length is increased by successive steps of, say, 0.10 Å., and wedge readings are repeated in the same manner at each change of wave-length, until at about 0.65 Å. from the centre of the

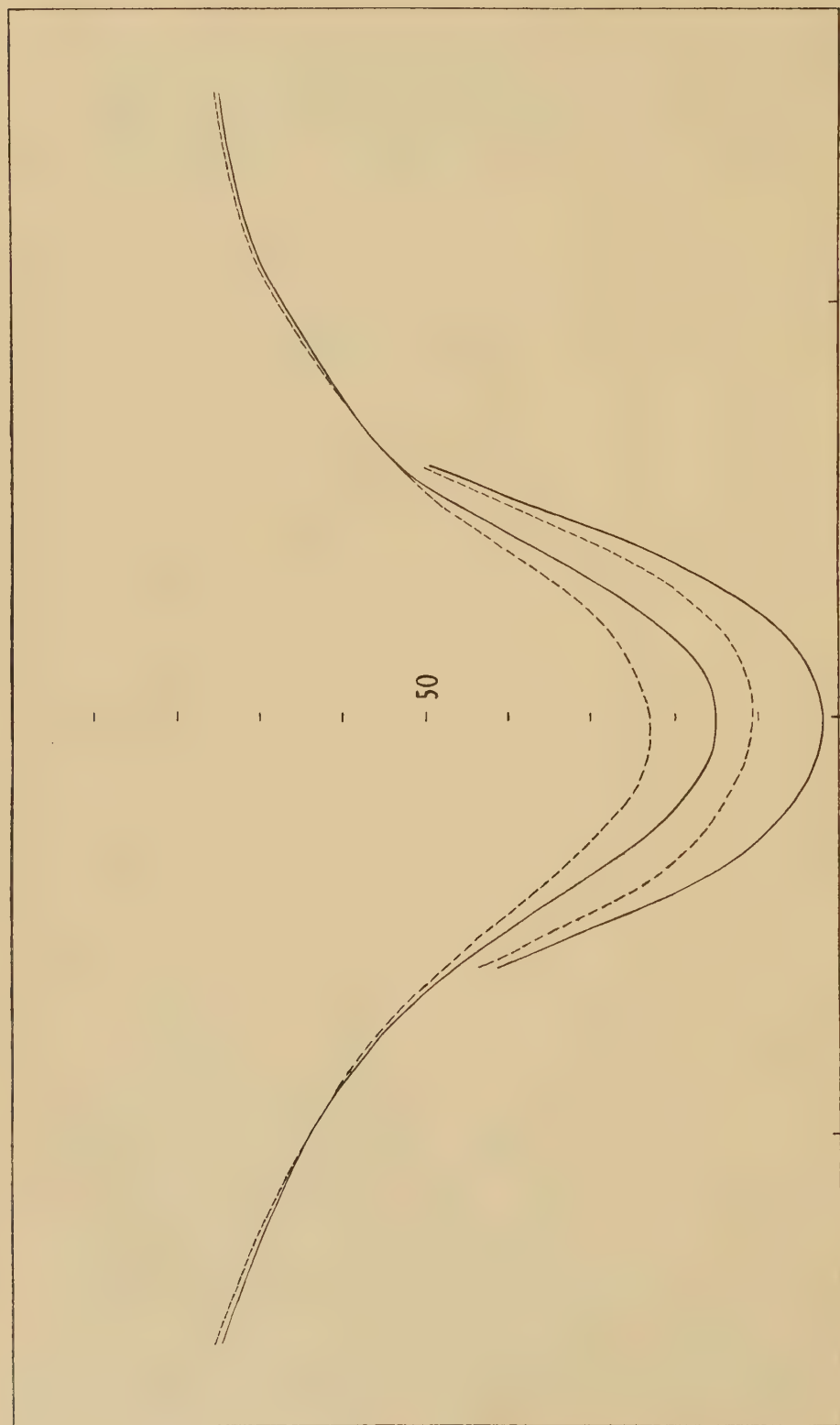
TABLE II

Contour of H α in Undisturbed Disk and in Dark Markings

Distance in Angstroms from Centre	Disk		Dark Marking	
	Uncorrected	Corrected	Uncorrected	Corrected
+4.0	95	95	95	95
3.0	87	87	87	87
2.0	78	78	78	78
1.5	76	75	76	75
1.0	68	67	68	67
0.9	65	64	65	64
0.8	61	61	61	61
0.7	58	58	58	58
0.6	53	53	50	50
0.5	48	46	38	36
0.4	40	37	28	23
0.3	32	27	20	14
0.2	27	20	14	5
+0.1	25	17	12	3
0.0	23	15	11	2
-0.1	24	16	12	3
0.2	26	19	14	6
0.3	30	25	18	10
0.4	36	32	24	18
0.5	42	39	32	29
0.6	48	46	44	42
0.7	53	52	53	53
0.8	57	57	57	59
0.9	61	60	61	61
1.0	64	64	64	64
1.5	76	74	76	74
2.0	(78)	(78)	(78)	(78)
3.0	86	86	86	86
-4.0	92	92	92	92

H α line the dark marking merges into the adjacent background (the wings of the *H α* line). The wedge readings when reduced give the ratio of intensity of background to dark marking; the observed intensity of the background at distances 0.10, 0.20 . . . 4.0 Å. from the centre of *H α* is known from the previously determined contour of *H α* observed with the same instrument by means of a lamp photometer (*M.N.*, 96, 8, 1935). Hence the contour of the dark marking can be inferred with respect to the continuous spectrum (=100) adjacent to the *H α* line.

This differential method has the advantage that the comparison between flocculus and background is made directly without the intermediary of a lamp. Use can therefore be made of short spells of fairly clear skies which



Distance from centre. $-1A.$ 0 $+1A.$ Distance from centre.

FIG. 2.—Contour of normal Fraunhofer line (upper pair) and dark marking (lower pair) shown as observed (---) and corrected (—). It will be noticed that the equivalent breadth is actually increased by the correction for heterochromatic light. The contour for the dark marking is not continued beyond $0.6 A.$ on either side of the centre, as the differences from the normal contour are too small to show clearly in the diagram.

are not quite steady enough in transparency or of sufficient duration to enable the lamp photometer to be used.

The disadvantage of this method is that the reduction of the contour of the dark marking depends on the previously determined contour of the normal $H\alpha$.

Table II and fig. 2 show the contour of $H\alpha$ as given in our previous paper, and also a mean contour for dark markings, constructed from observations of five different markings. The contours are given both uncorrected

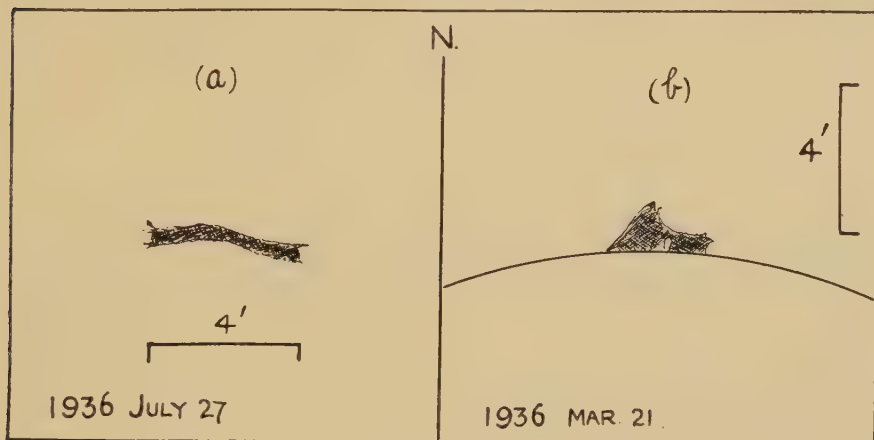


FIG. 3.

and corrected according to the methods described in paragraphs 2 and 3, with $h=2.5$ and $s=0.05$.

The dark flocculi giving the typical contour in Table II were observed on 1936 April 18 (two markings), May 4, May 6 and July 27. One of them is represented in fig. 3 (a). They were quite definite objects (necessarily so selected to make a difficult observation possible) and ranged in length from $2'$ to $4\frac{1}{2}'$ and from $\frac{1}{3}'$ to $1'$ in breadth. (The field of view of the spectro-helioscope is $4'$ wide, the height being adjustable by means of a stop.) For purposes of identification the following approximate positions of these five flocculi are given:—

Date 1936	Longitude	Distance from Central Meridian	Latitude
April 18	36°	2° W.	43° S.
18	62°	28° W.	30° N.
May 4	232°	51° W.	30° N.
6	152°	4° E.	44° N.
July 27	134°	18° E.	26° S.

A check on the order of accuracy of the contour of $H\alpha$ given in Table II by these five sets of observations is provided by three other sets of measures made by Newton and Barton with the lamp photometer on May 18, August 23 and August 24 respectively. This second method of obtaining the contour is exactly the same as that used in obtaining the contour of the normal $H\alpha$

line described by us in *M.N.*, **96**, 8, 1935, except that in this case the observer views the portion of the dark flocculus selected for measurement against its chromospheric background and matches it with the comparison strip given by the lamp beam which passes through the adjustable wedge. The observation is rather difficult to make and demands a steady sky that occurred only rarely at Greenwich during the summer of 1936. There is rather a scatter shown by the observations, but the following mean values (uncorrected) are in reasonable agreement with those given in Table II by the direct wedge method:—

Distance from Centre	<i>I</i> (Continuous Spectrum = 100)
·00 A.	11·5
·31 A.	20
·61 A.	40
·70 A.	52

It will be seen from Table II that the corrected central intensity of the dark markings is 2 per cent. of the continuous background outside the line. About half the correction from $r = \cdot 11$ to $r = \cdot 02$ is due to finite resolving power and half to heterochromatic light. An alternative correction with $h = 3$ and $s = 0\cdot 03$, which also gives $0\cdot 15$ for the central intensity on the undisturbed disk, gives the same value $0\cdot 02$ for the dark marking. One of us (H. W. N.) has the impression that there are narrow markings darker than those described here, but it is very difficult to form an opinion on such a matter where contrast is everything. On the whole we are inclined to assign a probable error of $\pm 0\cdot 01$ to the central intensity $0\cdot 02$, but to admit that there may be an over-correction of $0\cdot 01$ or $0\cdot 02$ —solely on the ground that other markings look darker. No doubt the quantity varies from one dark marking to another. As regards the *width* of the contour of the dark markings, it may be noted that measures made at Greenwich during 1936 show variations in the range of wave-length over which the marking is visible in the spectrohelioscope. For eleven markings with a mean radial velocity of ± 44 km./sec. the measured range was $0\cdot 93$ A. on either side of the maximum intensity, while for forty-four markings with a mean velocity of $\pm 1\cdot 5$ km./sec. the corresponding quantity was $0\cdot 62$ A.

5. *Contour of a Typical Stable Prominence.*—The following measures (Table III) show the mean contour of a number of typical stable prominences of $1'$ to $2'$ in height above the limb. They were measured with the lamp photometer, and the unit of intensity is the light of the continuous spectrum just outside $H\alpha$ at the *centre* of the disk. The observations giving the mean contour represented in Table III were obtained on 1936 February 13, February 20, March 18, March 21, May 19, May 20 and June 17, the respective position angles of the prominences (measured from the north point through E., S., W. and N.) being approximately 110° , 19° , 184° , 9° , 131° , 101° and 146° . Fig. 3 (*b*) shows the type of prominence measured. The contour was usually determined on one side only of the centre of $H\alpha$; the results have been meaned to give one side and the contour has been assumed to be symmetrical.

TABLE III

Contour of Typical Stable Prominence

Distance from Centre in Angstroms	Intensity	
	Uncorrected	Corrected
1.0	0	0
0.9	1	1
0.8	1	1
0.7	2	2
0.6	3	3
0.5	5	5
0.4	6	6
0.3	7	7
0.2	8	9
0.1	9	11
0.0	10	12

Other prominences have been observed at Greenwich during 1935-36 which differed from this mean prominence markedly. Some are, for example, about twice as bright or greater; one of eruptive type having a measured intensity of the maximum ordinate of brightness, displaced by Doppler effect, of nearly 60 per cent. (uncorrected) that of the continuous spectrum 10 Å. from $H\alpha$ at the centre of the disk. The mean intensity at the maximum ordinate of brightness given by 61 stable prominences during 1936 is, however, 9 per cent. (continuous spectrum = 100), so that the contour given in Table III may be regarded as an average one for this class of prominence that appears to produce absorption markings on the disk, the typical $H\alpha$ contour of which is given in paragraph 4.

6. *Discussion.*—Our measures show that the average stable dark marking emits less radiation per square second of arc than the average stable prominence, in the wave-length of the centre of $H\alpha$. (Outside the line, of course, the reverse must be the case.) We are aware of a possibility of systematic error, since the observer would tend to pick out the brighter parts of a prominence and the darker parts of a dark marking in considering the object—a dark part of a prominence would be considered as a hole through it. Nevertheless the objects which we have described here were all more or less structureless.

It might be thought that if the hydrogen cloud constituting a prominence when seen at the limb and a dark marking when seen on the disk is emitting radiation isotropically, there is no reason why the prominence (side view of the cloud) should appear brighter than the dark marking (top view of the cloud). There is, however, the consideration that markings vary in intensity according to the absorbing power of the cloud, that is to say, presumably according to its density or thickness; but the denser markings certainly do not give rise to weaker prominences. There is a definite tendency for prominences to give no markings on the disk at solar minimum.*

The observed phenomena seem to be explained by the view that the

* We owe these remarks to Mr. Evershed.

hydrogen cloud does not emit radiation from its interior, but behaves very like an opaque body. The depth of the contour of the dark marking shows that the cloud has an optical depth of 4 or 5 in the wave-length of the centre of $H\alpha$; accordingly the radiation field at the side of the cloud exposed to light from the photosphere is controlled almost entirely by inflowing radiation. This inflowing radiation is selectively scattered, and a certain percentage is scattered backwards, the process being very like reflection of sunlight from the surface of a thin cloud in the sky—except that in the solar cloud the process is monochromatic.

A simple model of the prominence and dark marking may be made by sticking a small tuft of cotton wool to the surface of an opaque 60-watt lamp, and turning the lamp slowly. When the cotton is at the “limb” it projects as an illuminated object like a prominence, and on turning the lamp so as to bring the cotton on to the “disk” it resembles a dark marking. In this case, as in the solar case, the side view of the illuminated projection is brighter than the top view (view on the disk) where the eye perceives light which has had to struggle through the opaque object.

Besides accounting for the appearance of the hydrogen cloud as a prominence outside the limb, and when it is half over the limb, and for the observation that the cloud appears darker near the centre of the disk than when seen at the limb, the view that the radiation from a hydrogen cloud is a cloud reflection of outside radiation accounts for the observed ratio of the intensities of prominences in $H\alpha$ and $H\beta$ determined at Greenwich. If we take as unit the emission in 1 A. of the continuous spectrum from the centre of the disk just outside the line concerned, the ratio found from Greenwich observations of contours of $H\alpha$ and $H\beta$ emission from eight prominences observed during 1936 is 4.6. It has been shown by one of us * that a ratio of 4.4 in these units (differing very widely from the value appropriate to thermodynamical equilibrium at 6000°, namely, 11.4) is to be expected if the outside of the prominence is in polychromatic radiative equilibrium with the radiation from the photosphere, assuming a central intensity of 0.01 for the Lyman alpha line. Other observers, notably Schwarzschild and Perepelkin, have found Balmer decrements in prominences which deviate markedly from thermal equilibrium values; but Menzel and Cillie † find approximately equilibrium values for the hydrogen emission from the chromosphere at a height of 670 km. (at the 1932 eclipse). This difference between observations of chromosphere and prominence seems too great to be attributed to photometric error. ‡ On the reflection-opacity view of prominence radiation, there is no reason to suppose that the decrement from the chromosphere and prominences should be the same.

Summary.—Visual photometric observations made during 1936 of the intensities of stable dark markings and prominences in $H\alpha$ light are described.

* Woolley, *M.N.*, 96, 515, 1936.

† *Ap. J.*, 85, 88, 1937.

‡ (Note added in proof.) It must be remarked that observations made by A. D. Thackeray on plates taken at the eclipse of 1936 by no means support Menzel and Cillie's value for the ratio $H\alpha/H\beta$.

The observations have been carried out with auxiliary photometric apparatus attached to the Hale spectrohelioscope at Greenwich. Corrections to the contours of the line for finite resolving power and for heterochromatic light (scattered light) are introduced.

The central intensity (corrected) of the dark markings is 0.02, and the brightness of the (stable) prominences is 0.12, the unit being the intensity of the centre of the disk in the case of prominences and the adjacent disk in the case of dark markings, in both cases in wave-lengths just outside $H\alpha$. In a discussion it is suggested that the hydrogen clouds forming quiescent prominences are not self luminous, but opaque, so that the prominences are seen in reflected light, in a manner similar to a common cloud.

Mr. H. J. Barton has shared in the observations described above.

VARIABLES IN KAPTEYN'S SELECTED AREAS 2-19.

E. A. Baker, D.Sc.

(Communicated by Professor R. A. Sampson, F.R.S.)

I. Variables in Area 2

This paper is the first result of a systematic examination of the variables situated in the selected areas centred near 75° and 60° N. declination. The instruments and methods employed will be fully described in the new *Publications of the Royal Observatory, Edinburgh*, the first parts of which will be issued shortly.

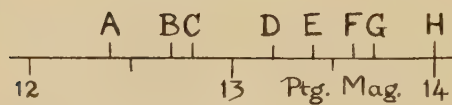
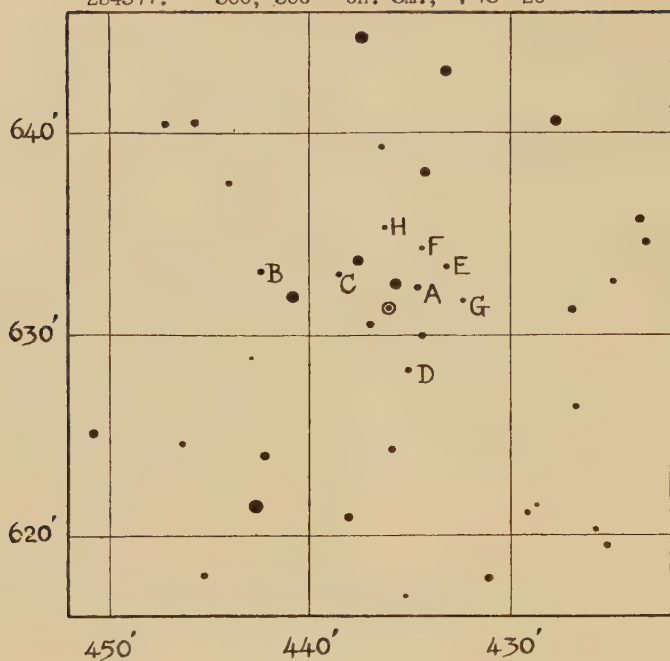
The aim of the work is to furnish reasonably complete information on all variables in the regions examined which rise above 14th magnitude (photographic). The photographic plates, taken with a triplet lens of 250 mm. aperture and 1.5 m. focal length, cover an area of about 40 square degrees, and with an exposure of 30 minutes reach about 16th magnitude. It is hoped to publish for each variable the following:—

1. Map of surrounding stars.
2. Photographic magnitudes of comparison stars.
3. Position correct to about 1 second of arc.
4. Observations sufficient to give maxima and minima and, in most cases, light curves.

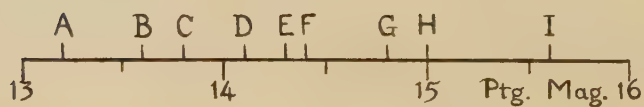
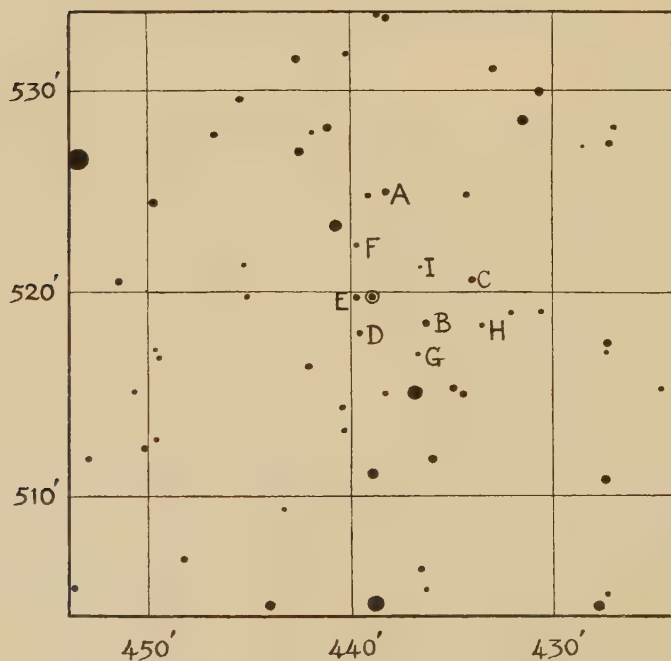
The magnitudes of the comparison stars are based on those of the *Mount Wilson Catalogue* of photographic magnitudes in the selected areas, while the positions for areas centred near $+75^\circ$ are by comparison with those given in the *Greenwich Astrographic Catalogue*, vol. iii. Exact positions, while not necessary, have been given, since in some cases known variables are listed with positions so rough as to be misleading.

In Area 2 the variables TX Cep, RV Cep, TZ Cep and YZ Cas were known before these observations were started. Of these, the last is an Algol type variable, which is too bright and varies over too small a range to be suited

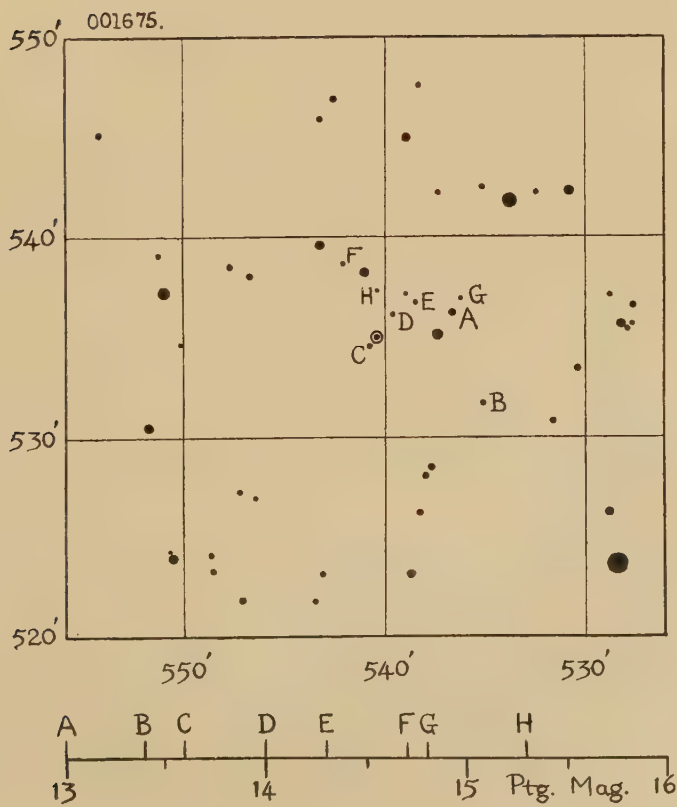
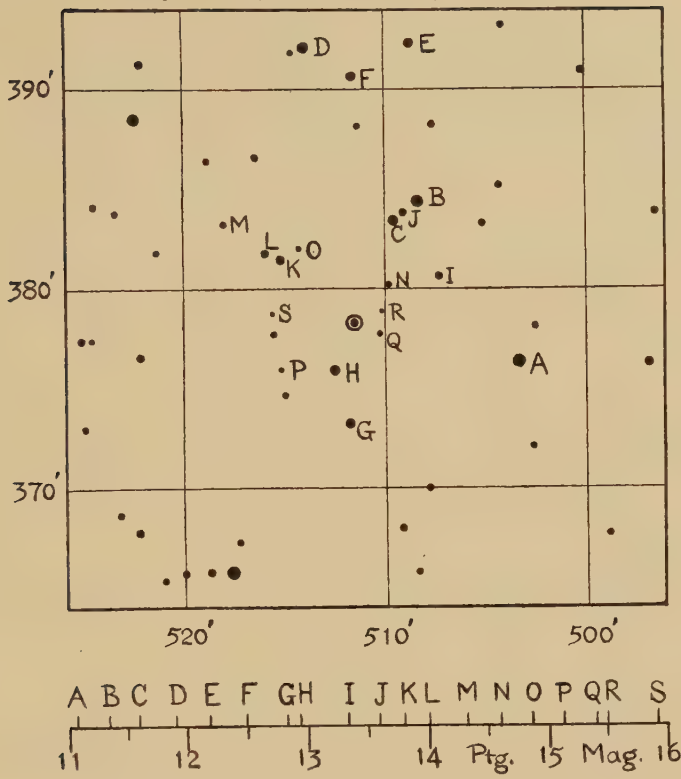
234577. 500, 500 = 0h. 5m., + 75° 20'



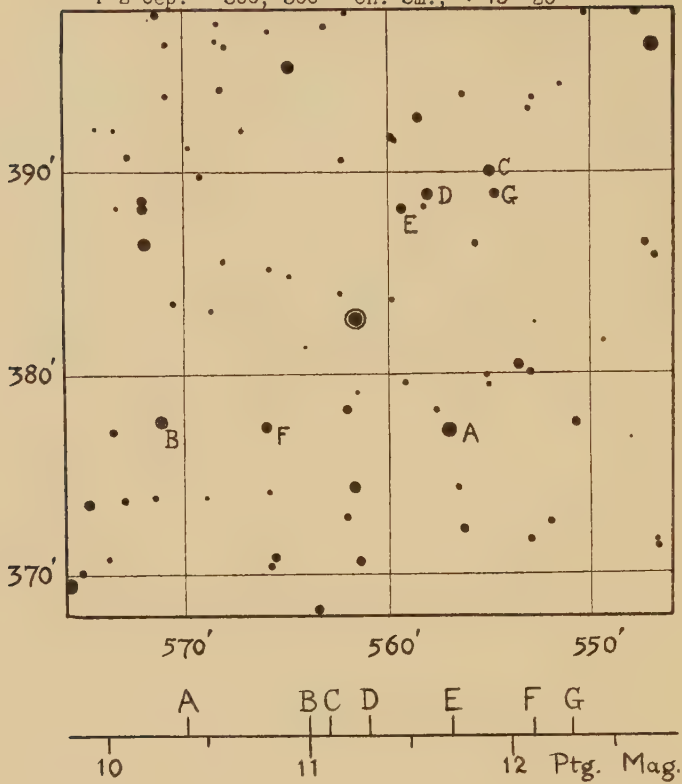
T X Cep.



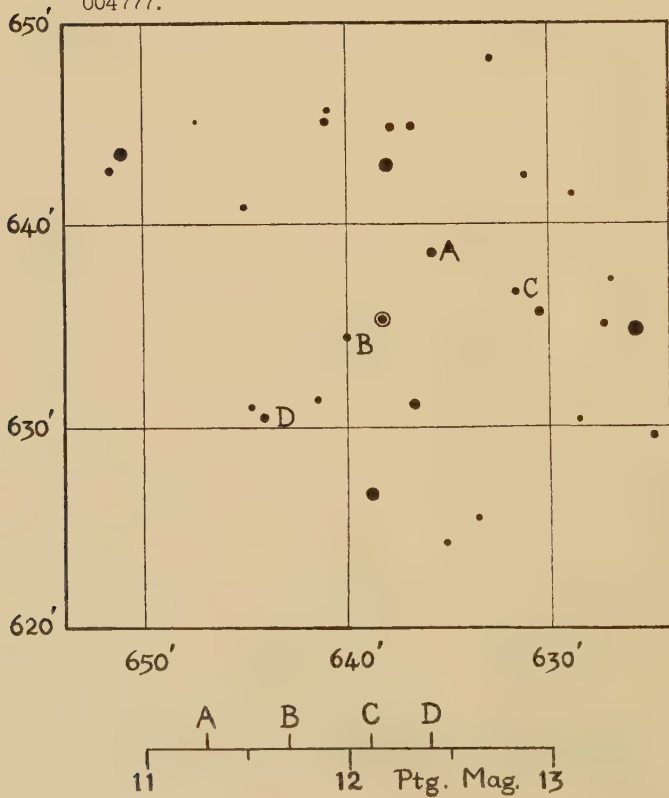
R V Cep. 500, 500 = 0h. 5m., + 75° 20'



T Z Cep. 500, 500 = 0h. 5m., + 75° 20'



004777.



to methods designed for faint variables. It has been the subject of numerous investigations and is therefore omitted here. To the remainder our observations have added three new variables, referred to below by their Harvard designations 234577, 001675 and 004777.

The maps show the regions surrounding the variables, with provisional magnitudes of the comparison stars. The table gives the observations based

Star	Unnamed	TX Cep	RV Cep	Unnamed	TZ Cep	Unnamed
Harvard designation	234577	234875	000773	001675	001973	004777
	h m s	h m s	h m s	h m s	h m s	h m s
R.A. 1900.0	23 45 19.7	23 48 36.4	0 7 41.4	0 16 1.7	0 19 23.0	0 47 20.8
Dec. 1900.0	77° 28' 43"	75° 37' 40"	73° 18' 26"	75° 53' 50"	73° 20' 52"	77° 22' 47"
Discovered	1936	1915	1906	1936	1921	1936
Period	94 ^d ?	380 ^d	272 ^d	125 ^d	84 ^d	? short
Ptg. max.	12.2	12-13	11.5	12.8	10.6	11.5
Ptg. min.	13.7	< 16	16	< 16	11.6	12.3
Ptg. mag. at J.D. 242						
7446.349	12.9	< 15	< 16	13.5	11.4	12.3
7453.418	12.5	16?	15.8	14.0	11.3	11.5
7474.392	12.9	...	14.4	14.7	10.8	11.5
7477.353	13.0	16?	14.1	< 16	11.0	11.5
7665.455	13.4	14.2	15.8	13.4	10.8	11.8
7719.375	13.2	14.3	14.4	14.5	11.0	12.2
7742.338	13.0	13.2	12.4	< 15.5	10.6	11.9
7836.390	13.3	15.5?	12.7	14.5	10.7	11.5
7859.426	13.7	15.5?	14.5	< 15.5	11.4	11.5
8036.353	13.2	< 15.5	11.8	15.5	11.6	11.5
8157.411	13.2	< 16	15.8	14.2	10.7	11.5
8164.545	13.1	16?	15.8	13.4	10.7	11.6
8184.413	13.4	< 15.5	15.8?	13.0	10.8	11.5
8190.413	13.3	< 15.5	15.8?	13.2	10.7	11.5
8210.344	13.3	< 15	15.8?	14.7	11.0	11.5
8449.332	12.2	15.5	< 15.5	13.6	11.4	11.6
8491.503	(< 12)	(< 12)	(< 12)	(< 12)	10.8	11.9
8541.456	13.1	< 14.8	< 15.7	13.2	10.7	11.5
8571.364	12.8	< 15	14.7	12.8	10.6	11.4
8574.347	12.4	< 15.5	13.7	13.2	10.6	11.5

on these assumed magnitudes. The following are notes to these observations :—

234577.—This is believed to be a semi-regular variable with a period of about 94 days; but the observations are not yet sufficiently numerous to make this period definite.

234875 (TX Cep).—Owing to the nearness of the period to one year the observations are not well scattered over the period. The magnitude at minimum was known to be below 14 (visual) and is now shown to be below 16 (photographic).

000773 (RV Cep).—The magnitude at minimum, previously not determined, appears from these observations to be 16 (photographic). Both at maximum and at minimum the magnitude appears to vary considerably.

001675.—This new variable is of the α Ceti type with a very regular variation, the present observations giving a satisfactory light curve.

001973 (TZ Cep).—This variable, owing to its brightness, has been much observed and discussed, and the present observations add little to the information now available from other sources.

004777.—This variable is probably of the eclipsing type, and is bright enough for visual observations.

The ruled co-ordinates on the maps are for each 10', the centre of the area ($0^h 5^m$, $+75^\circ 20'$, 1900.0) being given the co-ordinates 500, 500. This system resembles closely that used in the Bergedorfer Spektral-Durchmusterung. The fact that the top of the map is not precisely north does not seem a material objection to this convenient method of showing the results.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE UNIVERSITY OF LONDON OBSERVATORY IN THE YEAR 1936.

(Communicated by the Director)

This report concerning occultations observed and reduced at Mill Hill is the sixth of a series, of which No. 5 is in *M.N.*, 96, 709. Owing to an accident to R no observations were made before February 27, but subsequently all the occultations predicted in the *Nautical Almanac* were attempted. The two faint stars Nos. 119 and 120 were not predicted, and the proper motions in these cases are unknown.

From October 7 the six dots have been received on a tape chronograph made by the Cambridge Scientific Instrument Co., and given to the University of London Observatory by Mr. R. H. Fry of Barnet.

The error of the synchronome clock is found from the six-dot signals broadcast by the British Broadcasting Corporation—usually two or three comparisons before each occultation and two following.

Column 6 of Table I contains the actual U.T. of the observations. For the purpose of interpolating for parallax, right ascension and declination (but for no other purpose) $0^h.00152$ was added to the U.T. to correct for the error of 3" in the mean longitude of the Moon as observed by Emeritus Professor E. W. Brown. No other correction for lag, personal error, known irregularities in the contour of the Moon's limb, etc., has been included. Observations 117, 125 and 137 were made by G, using 6-inch, 6-inch and 24-inch. R used 8-inch always. The reductions were made by R in the manner prescribed by the Nautical Almanac Office and checked by G.

TABLE I

Serial No.	Date	Star	Mag.	Phase	U.T.	χ	$\cos(\chi - \rho)$	$\sin(\chi - \rho)$	$\sigma' - \sigma$
	1934				h m s				
115	Feb. 27	47 Arietis	5.8	D	18 51 54.02	91 30	+ .965	+ .263	+ 1.2
116	Mar. 3	209 B Geminorum	6.1	D	22 20 48.10	51 40	+ .592	- .808	- 1.2
117	Apr. 4	e Leonis	5.3	D	22 29 10.64	169 38	+ .572	+ .820	+ 1.6
118	4	e Leonis	5.3	D	22 29 10.69	169 38	+ .572	+ .820	+ 1.6
119	26	B.D. +21° 1564	8.1	D	20 34 39.24	46 46	+ .566	- .825	- 0.6
120	26	B.D. +20° 1768	8.4	D	21 42 42.33	69 49	+ .841	- .541	- 1.3
121	Sept. 9	1 Geminorum	4.3	Db	4 36 25.12	23 2	+ .312	- .950	- 3.6
122	9	1 Geminorum	4.3	R	5 2 28.84	342 3	- .390	- .921	+ 0.1
123	27	137 B Capricorni	6.2	D	20 18 53.82	61 43	+ .993	- .115	- 0.6
124	28	κ Aquarii	5.3	D	23 2 51.04	46 43	+ .941	- .338	- 0.6
125	28	κ Aquarii	5.3	D	23 2 51.37	46 43	+ .941	- .338	- 0.7
126	Oct. 3	63 Arietis	5.2	R	20 31 53.36	277 29	- .948	- .318	+ 1.7
127	3	65 Arietis	5.9	R	21 11 34.02	266 56	- .991	- .134	$\pm$ 0.0
128	5	T Tauri	4.3	Db	3 2 4.92	140 22	+ .593	+ .805	+ 3.7
129	5	T Tauri	4.3	R	3 44 10.99	209 6	- .532	+ .847	+ 0.1
130	6	394 B Tauri	6.1	R	3 21 47.96	292 30	- .902	- .431	- 1.8
131	31	A Tauri	4.5	Db	21 56 43.83	100 49	+ .951	+ .311	+ 1.3
132	31	A Tauri	4.5	R	22 55 44.14	233 16	- .866	+ .501	+ 1.0
133	Nov. 6	209 B Cancr	6.5	R	5 12 53.65	274 59	- .965	+ .262	+ 0.3
134	21	c ² Capricorni	6.2	D	16 40 29.95	18 28	+ .647	- .763	- 0.5
135	21	B.D. -9° 5854	6.8	D	19 6 5.02	83 17	+ .965	+ .263	+ 0.9
136	22	κ Aquarii	5.3	D	16 45 1.86	86 57	+ .941	+ .338	- 2.5
137	Dec. 16	57 Sagittarii	6.0	D	16 55 50.38	118 56	+ .720	+ .694	+ 0.9
138	30	209 B Cancr	6.5	R	22 44 28.58	289 58	- 1.000	+ .002	- 2.3

TABLE II

Co-ordinates of Stars when not given in the N.A. for 1936

Serial No.	α	$\Delta\alpha$	δ	$\Delta\delta$	Authority
	h m s	s	° ' "	"	
119	7 13 31.46	+ 1.79	+ 20 51 37.1	- 2.4	1097 Cat. 3553 Étoiles Brüssel.
					4199 Cat. 14263 Étoiles Abbazia.
120	7 16 1.50	+ 1.80	20 35 3.9	- 1.6	4232 Cat. 14263 Étoiles Abbazia.
126	3 19 3.98	+ 4.80	20 30 52.5	+ 17.7	A. E. } 1936.
127	3 20 49.27	+ 4.80	20 34 57.1	+ 17.5	A. E. }
130	5 39 26.32	+ 4.67	23 10 31.2	+ 9.1	A. E. }
133	9 6 18.00	+ 3.78	11 49 33.6	- 15.5	W. Z. C. (Hedrick).
137	19 48 28.94	+ 3.27	- 19 12 32.4	+ 10.2	Hammond MS.
138	9 6 18.00	+ 5.47	11 49 33.6	- 24.5	W. Z. C. (Hedrick).

OCCULTATIONS OF STARS BY THE MOON OBSERVED AT BIRMINGHAM UNIVERSITY DURING THE YEAR 1936.

James Young.

This report continues those of previous years.

Table I gives the usual particulars of the observations. The U.T.'s are uncorrected for lag. Notes are given at the end of the report concerning the conditions of observation in abnormal cases.

Table II gives the reductions. A correction of $0^{\text{h}}.00152$ has been added to the U.T.'s for the purpose of taking the Moon's right ascension, declination and parallax from the *Nautical Almanac*, as suggested by Professor E. W. Brown.

Table III gives the positions of the stars for which reductions are not given in the *Nautical Almanac*. The mean positions of Nos. 177 and 179 are taken from the *Nautical Almanac* for 1936, the others from the *Yale Photographic Catalogue*.

TABLE I

No.	Star	Mag.	Phase	Date	U.T.		
					d	h	m s
159	19 Psc	5.3	D	Jan. 00	18	06	41.2
160	B.D. +24° 1135	7.7	D	Feb. 03	18	30	51.8
161	B.D. +23° 1258	8.2	D	Feb. 03	20	35	29.9
162	B.D. +24° 1168	7.9	D	Feb. 03	20	36	13.4
163	8 Gem	6.1	D	Feb. 03	21	55	28.8
164	9 Gem	6.3	D	Feb. 03	22	15	43.9
165	o Leo	3.8	D	Feb. 07	21	00	09.1
166	o Leo	3.8	R	Feb. 07	22	09	32.8
167	209 B Gem	6.1	D	Mar. 03	22	17	49.5
168	B.D. +23° 738	8.3	D	Mar. 27	21	33	05.3
169	300 B Tau	6.2	D	Mar. 27	22	25	58.2
170	B.D. +20° 1846	8.9	D	Mar. 30	20	32	43.5
171	e Leo	5.1	D	Apr. 04	22	26	46.3
172	κ Aqr	5.3	D	Sept. 28	23	01	47.5
173	63 Ari	5.2	R	Oct. 03	20	32	49.9
174	65 Ari	5.9	R	Oct. 03	21	12	29.3
175	A Tau	4.5	D	Oct. 31	21	56	15.8
176	A Tau	4.5	R	Oct. 31	22	56	08.6
177	B.D. -14° 5908	6.6	D	Nov. 20	19	50	22.7
178	c ² Cap	6.2	D	Nov. 21	16	40	53.3
179	B.D. -9° 5854	6.8	D	Nov. 21	19	03	21.1
180	57 Sag	6.0	D	Dec. 16	16	51	16.1

TABLE II

No.	$\sigma' - \sigma$	χ	ρ	$\chi - \rho$	No.	$\sigma' - \sigma$	χ	ρ	$\chi - \rho$
	"	^o ' "	^o ' "			"	^o ' "	^o ' "	
159	-1.1	49 05	65 08	- 16 03	170	-0.7	116 32	103 49	+ 12 43
160	-0.7	28 27	95 29	- 67 02	171	+1.5	170 30	114 31	+ 55 59
161	-1.3	83 06	95 59	- 12 53	172	-1.7	42 51	66 30	- 23 39
162	-1.0	75 49	95 59	- 20 10	173	+0.6	280 00	79 06	-159 06
163	-1.0	61 29	96 19	- 34 50	174	+1.1	269 36	79 16	-169 40
164	+0.4	107 50	96 23	+ 11 27	175	-1.2	97 11	83 02	+ 14 09
165	-0.9	97 52	112 42	- 14 50	176	+1.7	237 09	83 18	+153 51
166	+0.7	314 01	112 49	-158 48	177	-0.2	64 40	70 17	- 5 37
167	-1.0	51 09	104 41	- 53 32	178	-1.0	14 48	68 11	- 53 23
168	-1.6	53 06	86 54	- 33 48	179	-0.2	79 23	68 00	+ 11 23
169	-1.1	174 01	87 08	+ 86 53	180	-0.1	113 30	74 58	+ 38 32

TABLE III

No.	$\alpha_{1936.0}$	$\Delta\alpha$	$\delta_{1936.0}$	$\Delta\delta$
	h m s	s	^o ' "	"
160	6 05 54.21	+2.77	+24 13 39.5	+ 1.6
161	6 10 14.46	+2.78	+23 58 25.2	+ 1.3
162	6 10 14.16	+2.78	+24 00 20.4	+ 1.3
168	4 40 29.39	+1.42	+23 57 56.3	+ 5.1
170	7 31 41.33	+2.33	+19 58 47.4	- 4.0
177	20 57 36.83	+3.60	-13 46 53.0	+17.1
179	21 47 10.95	+3.74	- 9 16 53.4	+21.6

Notes

159. Through thin cloud. Certain.
 165. Bright limb. Difficult.
 167. Through mist. Certain.

170. Faint. Difficult.
 175. Bright limb. Difficult.
 177. Through haze. Certain.

*The University,
 Edgbaston, Birmingham :
 1937 April 15.*

